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LOCALLY-STRUCTURED UNITARY NETWORK

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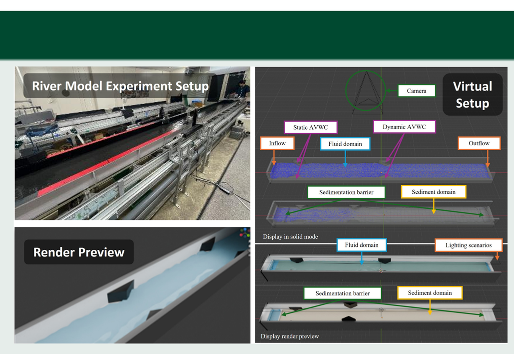
ABSTRACT

- Proposed a novel learnable linear transform, **Locally-Structured Unitary Network (LSUN)**, for interpretable and systematic manifold-based dimensionality reduction.
- LSUN employs **locally controllable, shift-variant filter kernels** under a global unitary constraint, trained in a self-supervised manner to capture tangent spaces efficiently.
- Demonstrated superior low-dimensional representation through approximation and dynamical system modeling, suggesting its potential as a **new paradigm for manifold learning**.
- **Key words** – *Tangent space sampling, shift variability, unitarity, linear transforms, self-supervised learning*

1 INTRODUCTION

Need for interpretable dimensionality reduction for high-dimensional data (e.g., observation of river-channel dynamics since 2017)



Problems

- High-dimensional data contain rich information
  $\longrightarrow$ Difficult to analyze due to the "curse of dimensionality"
- Conventional block processing and convolutional dictionary
  $\longrightarrow$ Unable to directly capture tangent-space structures
- Nonlinear approaches such as autoencoders or sparse coding
  $\longrightarrow$ Poor interpretability as an operator

Objective

Learnable linear transform that captures tangent spaces of a manifold latent in high-dimensional data

We propose **LSUN** to capture the tangent spaces.

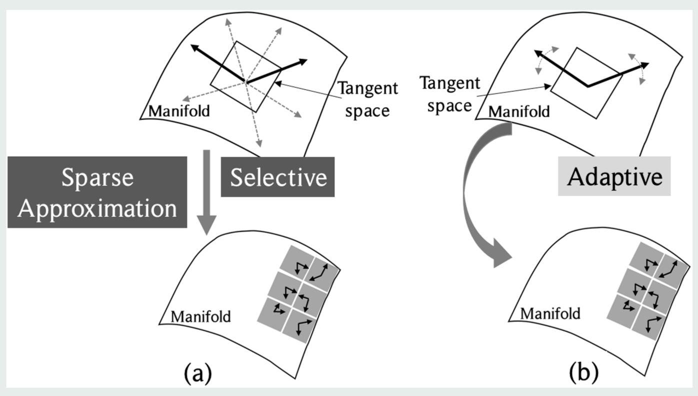


Figure: Two different ways of capturing tangent spaces

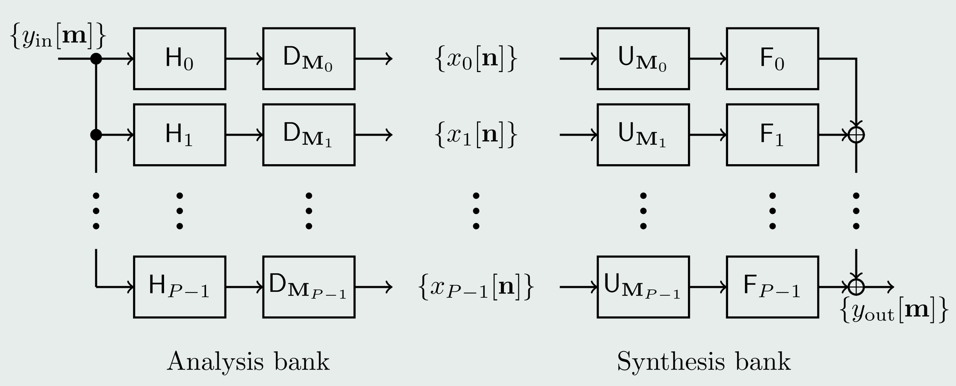


Figure: Parallel Config. of P-Ch. filter banks (FBs) w/ stride

Contributions

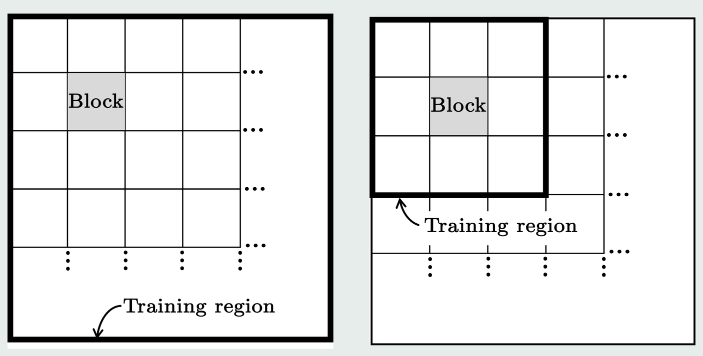
- ▶ **Shift-variability:** Adaptive filter kernels
- ▶ **Unitarity:** Energy preservation
- ▶ **Self-supervised learning:** Tunable w/o training data

2 REVIEW OF LINEAR DIMENSIONAL REDUCTION

Table: Comparison of dimensional reduction models.

	Unitary	Overlap	Learnable
Block DCT	✓	✗	✗
Block PCA	✓	✗	✓
CAE	✗	✓	✓
DL-PUFB (CDL)	✓	✓	✓

CAE: Convolutional Autoencoder, DL-PUFB: Dictionary Learning w/ Paraunitary FBs



(a) GBPCA (b) LBPCA

Figure: 2-D illustration of training region.

Global/Local Block PCA (GBPCA/LBPCA): The problem is set as

$$\{\hat{\mathbf{\Phi}}, \{\hat{\mathbf{x}}_n\}_n\} = \arg\min_{\{\mathbf{\Phi},\{\mathbf{x}_n\}_n\}} \frac{1}{2S}\sum_{n=1}^{S}\|\mathbf{y}_n - \mathbf{\Phi}\mathbf{x}_n\|_2^2,$$
$$\text{s.t.} \quad \mathbf{\Phi}^\intercal\mathbf{\Phi} = \mathbf{\Phi}\mathbf{\Phi}^\intercal = \mathbf{I}_M,$$
$$\|\mathbf{x}_n\|_0 \leq p, n \in \{1,2,\dots,S\}, p \in \{1,2,\dots,M-1\}$$

to find the unknown synthesis dictionary $\mathbf{\Phi} \in \mathbb{R}^{M\times M}$ and features $\{\mathbf{x}_n\}_n \subset \mathbb{R}^M$ for $\{\mathbf{y}_n\}_n$.

- GBPCA: $\mathbf{D} = \mathbf{A}^\intercal = \mathrm{blkdiag}(\mathbf{\Phi}, \mathbf{\Phi}, \cdots, \mathbf{\Phi})$ w/ $\mathbf{\Phi} \in \mathbb{R}^{M\times M}$
- LBPCA: $\mathbf{D} = \mathbf{A}^\intercal = \mathrm{blkdiag}(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \cdots, \mathbf{\Phi}_B)$ w/ $\mathbf{\Phi}_b \in \mathbb{R}^{M\times M}$

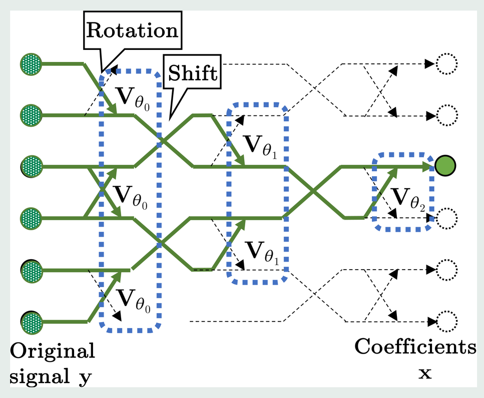
Convolutional Dictionary Learning (CDL): Let $\mathbf{D} \in \mathbb{R}^{BM\times BN}$ be a global synthesis convolutional dictionary, $\mathbf{x} \in \mathbb{R}^{BN}$ be a concatenation of coefficient vectors $\{\mathbf{x}_n\}_n$, and $\mathbf{y} \in \mathbb{R}^{BM}$ be a target signal. Then, we have

$$\{\hat{\mathbf{D}}, \hat{\mathbf{x}}\} = \arg\min_{\{\mathbf{D},\mathbf{x}\}} \frac{1}{2}\|\mathbf{y} - \mathbf{D}\mathbf{x}\|_2^2 \text{ s.t. } \|\mathbf{x}\|_0 \leq BK,$$

where $K \ll M$, i.e., $K \ll N$.

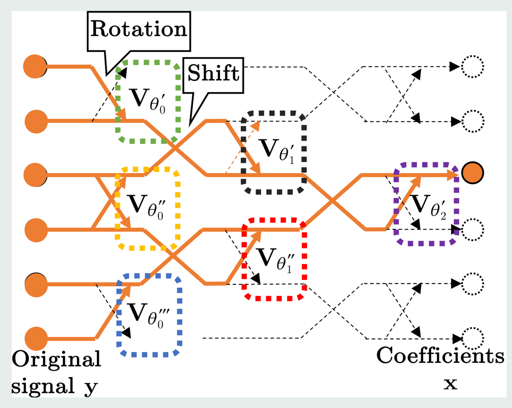
- Block LBPCA and Block K-SVD are special instances of CDL.

3 LOCALLY-STRUCTURED UNITARY NETWORK



Convolutional structure

Make filter kernels locally variable $\longrightarrow$ under the unitary structure



Shift-variant structure

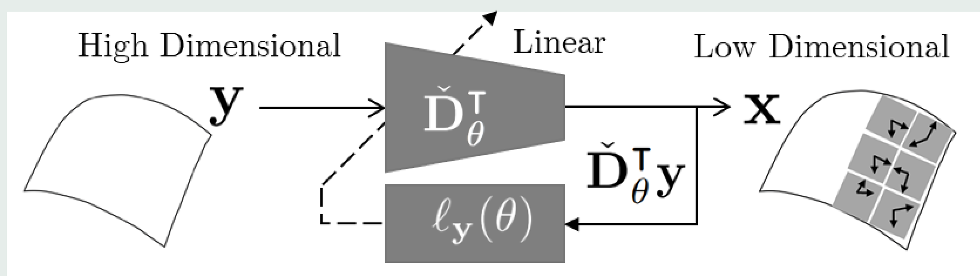


Figure: Example configuration of the learning process.

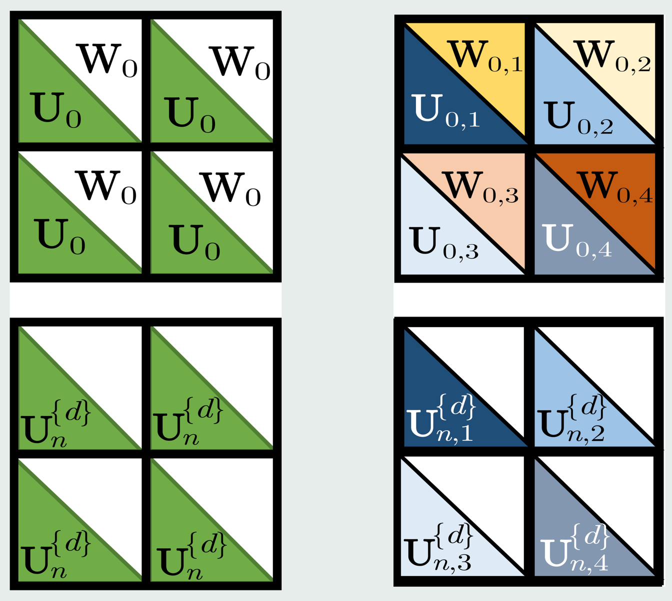
Minimize energy loss
$\ell_{\mathbf{y}}(\boldsymbol{\theta}) = \|\mathbf{y}\|_2^2 - \|\check{\mathbf{D}}_\theta^\intercal\mathbf{y}\|_2^2$
w/ a deep learning (DL) framework

4 CONSTRUCTION EXAMPLES

Based on convolutional FBs built on cascaded primitive block operations

1-D LSUN Example: Gan's 1-D PUFBs
L. Gan and K. K. Ma, "On Simplified Order-One Factorizations of Paraunitary Filterbanks", IEEE Trans. on Signal Process., March 2004

2-D LSUN Example: Our M-D LPPUFBs
S. Muramatsu, A. Yamada and H. Kiya, "A design method of multidimensional linear-phase paraunitary filter banks with a lattice structure", IEEE Trans. on Signal Process., March 1999



(a) Shift invariant (b) Shift variant

Figure: Comparison of unitary parameters

5 PERFORMANCE EVALUATION

Reproducible code $\rightarrow$ `https://github.com/msiplab/TanSacNet`

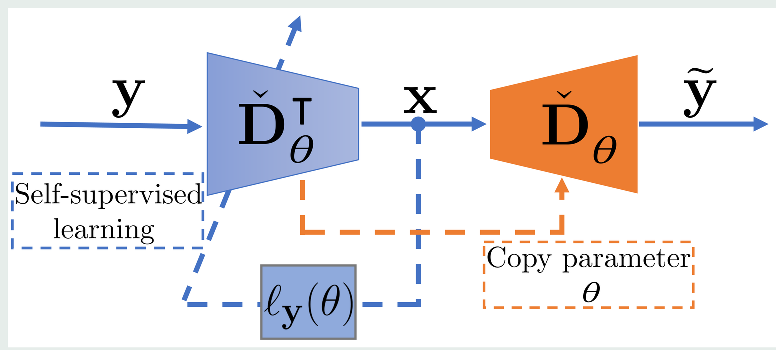


Figure: Example configuration of the learning process.

1-D LSUN Experiment: Function approximation. Please refer to the paper.

2-D LSUN Experiment: Image approximation. Please refer to the paper.

Dimensional Reduction for Dynamic System Modeling: Excerpt from the paper.

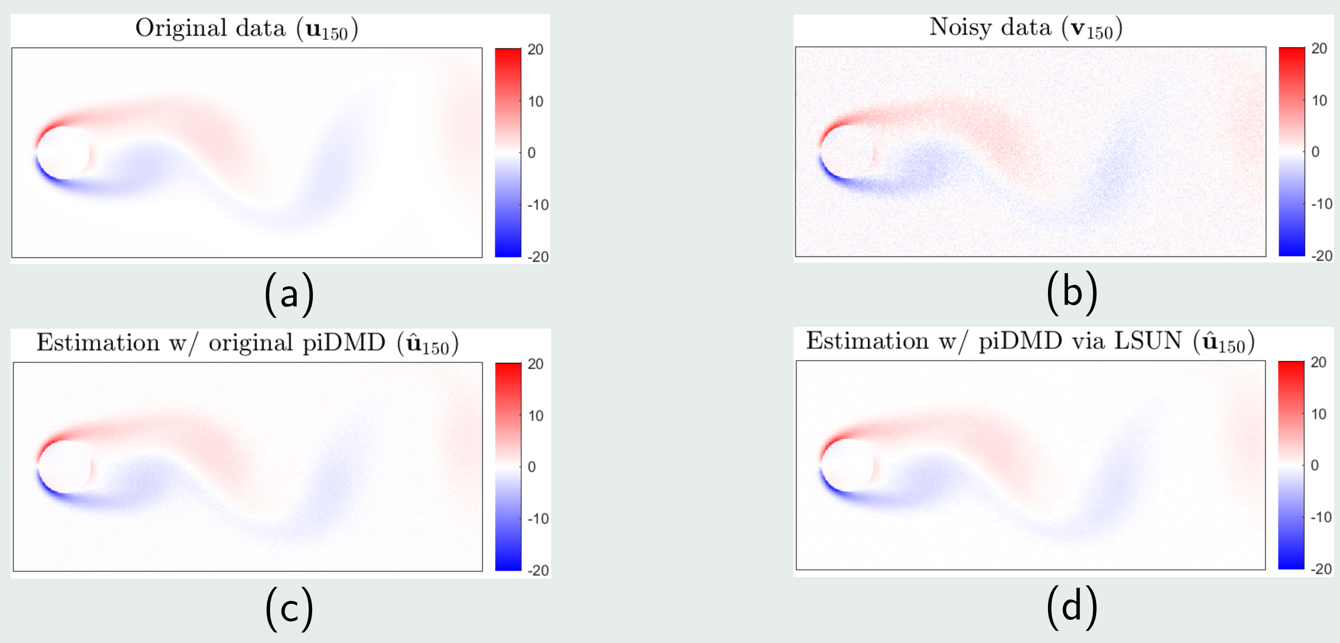
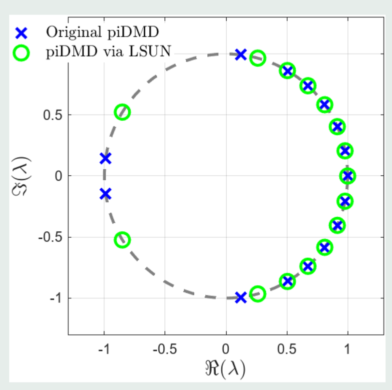
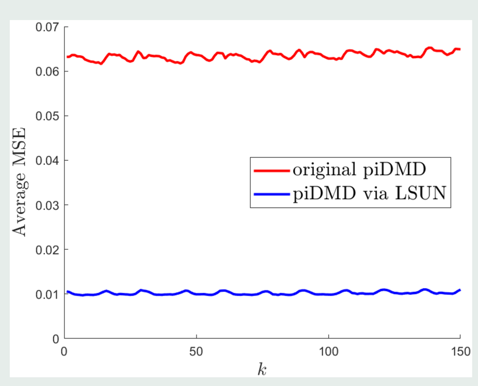


Figure: Flow estimation with noisy data using piDMD w/ and w/o LSUN.

piDMD: physics-informed dynamic mode decomposition, [Baddoo *et al.*, Proc. of the Royal Society, March 2023]





Realizing dimensional reduction w/ improved performance

6 CONCLUSIONS

- Proposed a locally-structured unitary dictionary for tangent space sampling, validated by approximation and dynamical system modeling
- Achieved better low-dimensional representations than existing block-based methods and convolutional dictionary learning methods
- Future work: apply LSUN to nonlinear conservative systems and integrate it into normalizing-flow networks for improved interpretability

Acknowledgment

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