Bifurcations from one-parameter families of symmetric periodic orbits in reversible systems

Kazuyuki Yagasaki
Mathematics Division, Department of Information Engineering, Niigata University, Niigata 950-2181, Japan
E-mail: yagasaki@ie.niigata-u.ac.jp

Abstract. We study bifurcations from one-parameter families of symmetric periodic orbits in reversible systems and give simple criteria for subharmonic symmetric periodic orbits to be born from the one-parameter families. Our result is illustrated for a generalization of the Hénon-Heiles system. In particular, it is shown that there exist infinitely many families of symmetric periodic orbits bifurcating from a family of symmetric periodic orbits under a general condition. Numerical computations for these bifurcations and symmetric periodic orbits are also given.

AMS classification scheme numbers: 37G15, 37G40, 34C23, 34C14, 37J15

1. Introduction

Reversible systems are frequently encountered in applications such as mechanics, fluids and optics, and have attracted much attention [11]. One of their characteristic properties is that periodic orbits are typically symmetric and appear in one-parameter families, in contrast to the fact that periodic orbits are typically isolated in general systems.

In this paper we study bifurcations from such one-parameter families of symmetric periodic orbits in reversible systems. We give simple criteria for “subharmonic” symmetric periodic orbits to be born from the one-parameter families. Here the word “subharmonic” means that the period of the new periodic orbit is given by $T = nT_0$ with $n \in \mathbb{N}$, where $T_0$ represents the period of the original periodic orbit. This is also contrasted to the fact that such bifurcations, which are called pitchfork or period-doubling bifurcations, commonly occur in general systems only for $n = 1$ or 2. Subharmonic bifurcations of symmetric periodic orbits in reversible systems were studied by Vanderbauwhede [20, 21] much earlier although such a simple criterion as obtained here was not given. Similar behavior for symplectic maps was also studied in [14]. See also [5, 6, 15].

Our result is illustrated for a generalization of the Hénon-Heiles system [10]. In particular, it is shown that there exist infinitely many families of symmetric periodic orbits bifurcating from a family of symmetric periodic orbits under a general condition. Two special cases are computed in details and bifurcation points are detected. Numerical
computations by a computer tool called AUTO97 [8] are also given. Standard pitchfork and period-doubling bifurcations, which are special ones of the subharmonic bifurcations treated here, in a similar system were studied and their relationship with the Lamé equation [23], which will be described in Section 4, were pointed out in [2, 3] earlier.

Our theory is presented in such a form as the Melnikov theory for subharmonics in forced oscillations [9, 24] since both of them treat “subharmonic” periodic orbits and are based on approximation of solutions on Gronwall’s lemma and the implicit function theorem (see Section 3). In particular, the criterion for bifurcations is stated with a matrix-valued function, which we will also call the Melnikov matrix or Melnikov function. However, both the theories are absolutely distinct. In particular, a different idea related to a fundamental characteristic of reversible systems (see Proposition 2.1) is essentially used and the Melnikov function is defined in a different manner without an integral (see Eqs. (6) and (13) below). Similarity between both the results may also be interesting. Moreover, many higher subharmonic bifurcations are detected by our theory and easily captured in numerical computations (see Section 4), in contrast with higher subharmonics detected by the Melnikov theory.

The outline of this paper is as follows. In Section 2 we state our problem and describe some fundamental properties of reversible systems in our setting. We present our main results and their proofs in Section 3. Our result is illustrated for the generalization of Hénon-Heiles system and numerical computations are given in Section 4.

2. Setup

We consider $2N$-dimensional systems of the form
\begin{equation}
\dot{x} = f(x), \quad x \in \mathbb{R}^{2N},
\end{equation}
where $N \geq 2$ is an integer and $f : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is $C^r$ ($r \geq 2$). We make the following assumptions.

(A1) The system (1) is reversible, i.e., there exists a (linear) involution $R : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ such that
\begin{equation}
f(Rx) + Rf(x) = 0 \quad \text{for all } x.
\end{equation}
Moreover, $\dim \text{Fix}(R) = N$, where $\text{Fix}(R) = \{x \in \mathbb{R}^{2N} \mid Rx = x\}$.

A fundamental characteristic of reversible systems is that if $x(t)$ is a solution, then so is $Rx(-t)$. We say that a solution $x(t)$ (and the corresponding orbit) is symmetric if $x(t) = Rx(-t)$. Moreover, an orbit is symmetric if and only if it intersects the space $\text{Fix}(R)$ [11, 22]. Note that $x = 0 \in \text{Fix}(R)$.

(A2) There exists a one-parameter family of symmetric periodic orbits, $x^\alpha(t)$, $\alpha \in \mathcal{A}$, with period $T_\alpha > 0$, in (1), where $\mathcal{A} (\neq \emptyset)$ is an open interval of $\mathbb{R}$ and $T_\alpha$ is bounded. Moreover, the following conditions hold:
(i) $x^\alpha(t)$ is $C^r$ with respect to $\alpha$ as well as $t$;
(ii) the two vectors
\[ \frac{\partial x^\alpha}{\partial t}(t) = f(x^\alpha(t)), \quad \frac{\partial x^\alpha}{\partial \alpha}(t) \]
are linearly independent for $t \in [0, T^n]$ and $\alpha \in \mathcal{A}$.

Note that $x^\alpha(t)$ intersects $\text{Fix}(R)$ but does not lie on $\text{Fix}(R)$, since if so, then $x^\alpha(t) = Rx^\alpha(-t) = x^\alpha(t)$ for any $t > 0$ in contradiction to $T^n > 0$. We set $x^\alpha(0) \in \text{Fix}(R)$ without loss of generality. Let
\[ \mathcal{N} = \{ x^\alpha(t) \mid t \in \mathbb{R}, \alpha \in \mathcal{A} \} \]
and
\[ \mathcal{N}_0 = \{ x^\alpha(0) \mid \alpha \in \mathcal{A} \} \subset \mathcal{N} \cap \text{Fix}(R). \]

We also notice that, by assumption (A1), there exists a splitting $\mathbb{R}^{2N} = \text{Fix}(R) \oplus \text{Fix}(-R)$, and we can choose a scalar product “.” such that
\[ \text{Fix}(-R) = \text{Fix}(R)^\perp. \quad (3) \]
Let $z_0(\alpha) = \dot{x}^\alpha(0)/|\dot{x}^\alpha(0)|$ and set $Z(\alpha) = \text{span}\{z_0(\alpha)\}$. Note that $z_0(\alpha)$ is $C^r$ in $\alpha$. Define a $(2N - 2)$-dimensional space $\bar{Z}(\alpha)$ by
\[ \mathbb{R}^{2N} = Z(\alpha) \oplus T_{x^\alpha(0)}\mathcal{N}_0 \oplus \bar{Z}(\alpha), \]
where the decomposition is also assumed to be orthogonal. See Fig. 1. We remark that $Z(\alpha) \subset \text{Fix}(-R)$ since by $x^\alpha(0) \in \text{Fix}(R)$
\[ R\dot{x}^\alpha(0) = R(f(x^\alpha(0))) = -f(R(x^\alpha(0))) = -f(x^\alpha(0)) = -\dot{x}^\alpha(0). \]

The following proposition is simple and easily proved but play a key role in our result below.

**Proposition 2.1.** Let $x(t)$ be a solution of (1). If it intersects $\text{Fix}(R)$ at $t = 0, T$ for some $T > 0$ but does not on $(0, T)$, then $x(t)$ is a symmetric $T$ - or $2T$ -periodic orbit.
Proof. Since it intersects $\text{Fix}(R)$ at $t = 0, T$, $x(t)$ is symmetric and $x(-T) = Rx(T) = x(T)$. Hence, $x(t)$ is $2T$-periodic if $x(0) \neq x(T)$ and it is $T$-periodic if $x(0) = x(T)$. \qed

This fact was repeatedly used in [18, 19] although it was not so clearly stated there. A homoclinic version of this result was also used as the key idea in [25].

3. Main Results

3.1. General Case

Let $\varepsilon$ be a small constant such that $0 < \varepsilon \ll 1$, and let $T > 0$ be a constant such that $Tn < T$ for all $n \in \mathcal{N}$. Using a slight modification of the proof of Lemma 4.5.2 of [9] based on Gronwall’s lemma, we obtain the following estimate for orbits near $x = x^\alpha(t)$.

**Lemma 3.1.** For $\varepsilon > 0$ sufficiently small, an orbit $x^\alpha_\varepsilon(t)$ of (1) passing through $x = x^\alpha(0) + \mathcal{O}(\varepsilon)$ on $\text{Fix}(R)$ at $t = 0$ can be expressed as

$$x^\alpha_\varepsilon(t) = x^\alpha(t) + \varepsilon w^\alpha(t) + \mathcal{O}(\varepsilon^2)$$

uniformly in $t \in [0, T]$ for all $\alpha \in \mathcal{A}$, where $w^\alpha(t)$ is a solution of the variational equation (VE) along $x = x^\alpha(t)$,

$$\dot{w} = Df(x^\alpha(t))w.$$  

Let $\Phi^\alpha(t)$ be the fundamental matrix of (5) with $\Phi^\alpha(0) = \text{id}_{2N}$, where $\text{id}_n$ is the $n \times n$ identity matrix. Then the solution of (5) with an initial condition $w(0) = w_0$ is written as $w = \Phi^\alpha(t)w_0$ for any $w_0 \in \mathbb{R}^{2N}$. Since $\dot{x}^\alpha(t)$ and $\partial x^\alpha(t)/\partial \alpha$ are solutions to (5), $Z(\alpha)$ and $T_{x^\alpha(0), \mathcal{N}}$ are invariant under the action of $\Phi^\alpha(T^n)$.

Let $z_j(\alpha), \tilde{z}_j(\alpha), j = 1, \ldots, N - 1$, be unit vectors which are $C^r$ in $\alpha$, such that

$$\tilde{Z}(\alpha) \cap \text{Fix}(-R) = \text{span}\{z_1(\alpha), \ldots, z_{N-1}(\alpha)\},$$

$$\tilde{Z}(\alpha) \cap \text{Fix}(R) = \text{span}\{\tilde{z}_1(\alpha), \ldots, \tilde{z}_{N-1}(\alpha)\}.$$

Note that $z_j(\alpha) \cdot \tilde{z}_l(\alpha) = 0$ for $j, l = 1, \ldots, N - 1$. Let

$$M^\alpha_{jl}(\alpha) = z_j(\alpha) \cdot \Phi^\alpha(nT_n)\tilde{z}_l(\alpha), \quad j, l = 1, \ldots, N - 1$$

for $n \in \mathbb{N}$ and define $M^\alpha(\alpha)$ as an $(N - 1) \times (N - 1)$ matrix whose $j\ell$-component is $M^\alpha_{jl}(\alpha)$. We call $M^\alpha(\alpha)$ the $n$-th order Melnikov matrix.

**Proposition 3.2.** Suppose that $\text{det} M^\alpha(\alpha)$ has a zero with rank $M^\alpha(\alpha) = N - 2$ at $\alpha = \alpha_0$, and that

$$\text{det} \begin{pmatrix} \frac{dM^\alpha}{d\alpha}(\alpha_0) \chi_0 & M^\alpha(\alpha_0) \\ 0 & \chi_0^T \end{pmatrix} \neq 0,$$

where $\chi_0 \neq 0 \in \mathbb{R}^{N-1}$ is an eigenvector of $M^\alpha(\alpha_0)$ for the zero eigenvalue. Then there exist a one-parameter family of symmetric periodic orbits which bifurcates from the family $\mathcal{N}$ at $\alpha = \alpha_0$ in (1). Moreover, the periods of the periodic orbits born at $\alpha = \alpha_0$ tend to $nT_{\alpha_0}$ or $2nT_{\alpha_0}$ as $\alpha \to \alpha_0$. 

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Proof. Assume that \( M^n(\alpha) \) satisfies the hypotheses. We can take a unit vector as \( \chi_0 \) without loss of generality and have

\[
M^n(\alpha_0)\chi_0 = 0, \quad \text{i.e.,} \quad \sum_{l=1}^{N-1} M^n_{jl}(\alpha_0)\chi_{l0} = 0 \quad \text{for } j = 1, \ldots, N - 1,
\]

where \( \chi_0 = (\chi_{10}, \ldots, \chi_{N-1,0})^T \in \mathbb{R}^{N-1} \) and the superscript "T" represents the transpose operator. Let \( x_\varepsilon^\alpha(t) \) be an orbit given by Lemma 3.1 for \( \varepsilon > 0 \) sufficiently small such that

\[
x_\varepsilon^\alpha(0) = x^\alpha(0) + \varepsilon \sum_{l=1}^{N-1} \chi_l \hat{z}_l(\alpha) \in \text{Fix}(R),
\]

where \( \chi = (\chi_1, \ldots, \chi_{N-1})^T \in \mathbb{R}^{N-1} \) is a unit vector. From Proposition 2.1 we see that if there exist \( \beta \in \mathcal{A}, T = nT_\alpha + O(\varepsilon) \) and \( \chi \) such that

\[
z_0(\alpha) \cdot x_\varepsilon^\alpha(T) = 0, \quad \frac{1}{\varepsilon} z_j(\alpha) \cdot x_\varepsilon^\alpha(T) = \sum_{l=1}^{N-1} M^n_{jl}(\alpha)\chi_l + O(\varepsilon) = 0, \quad |\chi|^2 - 1 = 0, \quad (9)
\]

then \( x_\varepsilon^\alpha(t) \) is a symmetric, \( T \)- or \( 2T \)-periodic orbit. Actually, if Eq. (9) holds, then \( x_\varepsilon^\alpha(T) \) is normal to \( z_j(\alpha) \) for \( j = 0, \ldots, N - 1 \), so that \( x_\varepsilon^\alpha(T) \in \text{Fix}(R) \).

It follows from (8) that Eq. (9) holds for \( (\alpha, T, \chi) = (\alpha_0, nT_{\alpha_0}, \chi_0) \) at \( \varepsilon = 0 \). The Jacobian matrix of (9) with respect to \( (\alpha, T, \chi, \varepsilon) = (\alpha_0, nT_{\alpha_0}, \chi_0, 0) \) is computed as

\[
\begin{pmatrix}
|x_\varepsilon^\alpha(nT_{\alpha_0})| & 0 & 0 \\
0 & \frac{dM^n}{d\alpha}(\alpha_0)\chi_0 & M^n(\alpha_0) \\
0 & 0 & 2\chi_0^T
\end{pmatrix},
\]

which is nonsingular by the second hypothesis. Here we have also used the fact that \( z_0(\alpha) \in \text{Fix}(-R) \) and \( x_\varepsilon^\alpha(T) = x^\alpha(0) \in \text{Fix}(R) \). Applying the implicit function theorem to (9), we obtain the result. \( \square \)

Remark 3.3. (i) For \( N = 2 \) the Melnikov matrix \( M^n(\alpha) \) is a scalar function and condition (7) is restated as

\[
\frac{dM^n}{d\alpha}(\alpha_0) \neq 0 \quad \text{(i.e., the zero } \alpha = \alpha_0 \text{ of } M^n(\alpha) \text{ is simple}).
\]

(ii) As stated above, \( Z(\alpha) \) and \( T_{x^\alpha(0)}N_\alpha \) are invariant under the action of \( \Phi^\alpha(T^\alpha) \).

Hence, if \( M^n(\alpha) \) satisfies the hypotheses of Proposition 3.2, then so does \( M^{kn}(\alpha) \) for any \( k \in \mathbb{N} \) since \( \Phi^\alpha(knT) = (\Phi^\alpha(nT))^k \).

We can refine Proposition 3.2 in the light of Remark 3.3(ii) as follows.

Theorem 3.4. Suppose that \( M^n(\alpha) \) satisfies the hypotheses of Proposition 3.2 and choose \( n \) to be the smallest of such positive integers. Let

\[
\hat{z}(\alpha) = \sum_{l=1}^{N-1} \chi_{l0} \hat{z}_l(\alpha).
\]
Then $\Phi^\alpha(t)\dot{z}(\alpha)$ is $nT_\alpha$- or $2nT_\alpha$-periodic only at $\alpha = \alpha_0$ in its neighborhood. Moreover, if the period of $\Phi^\alpha(t)\dot{z}(\alpha)$ is $nT_{\alpha_0}$ (resp. $2nT_{\alpha_0}$), then two branches (resp. one branch) of symmetric periodic orbits appear (resp. appears) from the family $\mathcal{N}$ at $\alpha = \alpha_0$ and the period of the periodic orbits tend to $nT_{\alpha_0}$ (resp. to $2nT_{\alpha_0}$) as $\alpha \to \alpha_0$. If $M^n(\alpha)$ has no zero at $\alpha = \alpha_0$ for any $n \in \mathbb{N}$, then such a family of symmetric periodic orbits does not exist.

**Proof.** We easily see by (2) and the symmetry of $x^\alpha(t)$ that

$$Df(x^\alpha(-t))Rw + RdDf(x^\alpha(t))w = 0$$

for all $w \in \mathbb{R}^{2N}$. It follows that if $w(t)$ is a solution of (5), then so is $Rw(-t)$. Assume that $M^n(\alpha)$ satisfies the hypotheses of Proposition 3.2. Then, only at $\alpha = \alpha_0$ near its neighborhood, $\Phi^\alpha(0)\dot{z}(\alpha), \Phi^\alpha(nT_\alpha)\dot{z}(\alpha) \in \text{Fix}(R)$. Hence, $\Phi^\alpha(t)\dot{z}(\alpha)$ is $nT_\alpha$- or $2nT_\alpha$-periodic, as in the proof of Proposition 3.2.

We turn to the second part. Note that $-\dot{z}(\alpha)$ may be taken instead of $\dot{z}(\alpha)$ in the proof of Proposition 3.2. Hence, if the period of $\Phi^\alpha(t)\dot{z}(\alpha)$ is $nT_\alpha$, then an $nT_\alpha$-periodic orbit is born from the periodic orbit $x^\alpha(t)$ for each of $\dot{z}(\alpha)$ and $-\dot{z}(\alpha)$. On the other hand, assume that the period of $\Phi^\alpha(t)\dot{z}(\alpha)$ is $2nT_\alpha$. Then $\Phi^\alpha(t)\dot{z}(\alpha)$ intersects $\text{Fix}(R)$ twice during one period. Hence, $\Phi^\alpha(nT_\alpha)\dot{z}(\alpha) = c\dot{z}(\alpha)$ for some $c \in \mathbb{R}$ by the invariance of $Z(\alpha)$. This means that $\Phi^\alpha(nT_\alpha)\dot{z}(\alpha) = c^2\dot{z}(\alpha)$, so that $c = -1$ since $\Phi^\alpha(nT_\alpha)\dot{z}(\alpha)$ is $2nT_\alpha$-periodic. Thus, only one $2nT_\alpha$-periodic orbit is born since the branches of solutions of (7) for $\dot{z}(\alpha)$ and $-\dot{z}(\alpha)$ correspond to the same branch of $2nT_\alpha$-periodic orbits.

Finally, we note that if $M^n(\alpha) \neq 0$ at $\alpha = \alpha_0$, then, near $(\alpha, T) = (\alpha_0, nT_{\alpha_0})$, the first equation of (7) does not hold and hence no symmetric periodic orbit of the form (4) exists. This means the last part.

**Remark 3.5.** From the proof of Theorem 3.4 we see that if $\Phi^\alpha(t)\dot{z}(\alpha)$ is neither $nT_\alpha$- nor $2nT_\alpha$-periodic for any $\chi_0 \in \mathbb{R}^{N-1}$, then $M^n(\alpha) \neq 0$.

### 3.2. Simple Case

Now we consider the special case of $N = 2$. Then the Melnikov matrix $M^n(\alpha)$ is a scalar function as stated in Remark 3.3(i). We also refer to $M^n(\alpha)$ as the $n$-th order Melnikov function.

Let $x = (\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$ and write $f(\xi, \eta) = (f_\xi(\xi, \eta), f_\eta(\xi, \eta))$ with $f_\xi, f_\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$. We assume the following additionally.

**(A3)** The two-dimensional $\xi$-plane is invariant under the flow generated by (1), i.e.,

$$f_\eta(\xi, 0) = 0 \text{ for all } \xi \in \mathbb{R}^2,$$

and the family $\mathcal{N}$ of periodic orbits is given by

$$\mathcal{N} = \{x^\alpha(t) = (\xi^\alpha(t), 0) \mid \alpha \in \mathcal{A}\},$$

i.e., it lies on the $\xi$-plane.
Obviously, $\xi = \xi^\alpha(t)$ satisfies
\[ \dot{\xi} = f_\xi(\xi, 0) \]
and
\[ D_\xi f_\eta(\xi, 0) = 0. \]
In addition, the $\xi$-plane is neither $\text{Fix}(R)$ nor $\text{Fix}(-R)$ since it intersects $\text{Fix}(R)$ and $x^\alpha(t) = (\xi^\alpha(t), 0)$ does not lie on $\text{Fix}(R)$. Since $Z(\alpha)$ and $T_{x^\alpha(0)}Q_0$ are contained in the $\xi$-plane, we can choose the $\eta$-plane as the space $\hat{Z}(\alpha)$ and take $z_1(\alpha) = (0, \zeta), \hat{z}_1(\alpha) = (0, \hat{\zeta}) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $\zeta, \hat{\zeta}$ are independent of $\alpha$ and satisfy $(0, \zeta) \in \text{Fix}(-R)$ and $(0, \hat{\zeta}) \in \text{Fix}(R)$.

Let $\Psi^\alpha(t)$ be the fundamental matrix of the normal variational equation (NVE) along $x = (\xi^\alpha(t), 0)$,
\[ \dot{v} = D_\eta f_\eta(\xi^\alpha(t), 0)v, \]
with $\Psi^\alpha(0) = \text{id}_2$. Substituting $z(\alpha) = (0, \zeta)$ and $\hat{z}(\alpha) = (0, \hat{\zeta})$ into (6), we obtain the following simpler form of $M^\alpha(\alpha)$:
\[ M^\alpha(\alpha) = \zeta \cdot \mu_\alpha \hat{\zeta}. \]
where $\mu_\alpha = \Psi^\alpha(T_\alpha)$ is the monodromy matrix of (12). In the present setting, we can restate Theorem 3.4 as follows.

**Theorem 3.6.** Suppose that $M^\alpha(\alpha)$ has a simple zero at $\alpha = \alpha_0$ and $n$ is the smallest of such positive integers. Then $\mu_\alpha \hat{\zeta} = \hat{\zeta}$ or $\mu_\alpha^2 \hat{\zeta} = \hat{\zeta}$ only at $\alpha = \alpha_0$ in its neighborhood. If $\mu_\alpha \hat{\zeta} = \hat{\zeta}$ (resp. $\mu_\alpha^2 \hat{\zeta} = \hat{\zeta}$), then two branches (resp. one branch) of symmetric periodic orbits appear (resp. appears) at $\alpha = \alpha_0$ and the period of the periodic orbits tend to $nT_\alpha$ (resp. to $2nT_\alpha$) as $\alpha \to \alpha_0$. If $M^\alpha(\alpha)$ has no zero at $\alpha = \alpha_0$ for any $n \in \mathbb{N}$, then such a family of symmetric periodic orbits does not exist.

4. Example

As an example, we consider
\[ \begin{align*}
\dot{\xi}_1 &= \xi_2, \\
\dot{\xi}_2 &= -\xi_1 - d\xi_1^2 - c_1\eta_1^2 - c_2\eta_1^3, \\
\dot{\eta}_1 &= \eta_2, \\
\dot{\eta}_2 &= -\omega^2\eta_1 - 2c_1\xi_1\eta_1 - 3c_2\xi_1\eta_1^2,
\end{align*} \quad (14) \]
which is a Hamiltonian system with Hamiltonian
\[ H(x, y) = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \omega^2\eta_1^2 + \eta_2^2) + \frac{1}{3}d\xi_1^3 + c_1\xi_1\eta_1^2 + c_2\xi_1\eta_1^3, \]
where $c_1, c_2$ and $d$ are constants. In particular, when $c_2 = 0$, Eq. (14) is the (generalized) Hénon-Heiles system, which was originally studied by Hénon and Heiles [10] for $c = 1, d = -1$ and $\omega = 1$. Henceforth we assume that $d = 1$ since we can set $d = 1$ by some change of coordinates if $d \neq 1$.

Equation (14) is reversible with respect to a linear involution
\[ R : (\xi_1, \xi_2, \eta_1, \eta_2) \mapsto (\xi_1, -\xi_2, \eta_1, -\eta_2) \quad (15) \]
and the $\xi$-plane is invariant. Moreover, there exists a one-parameter family of symmetric periodic orbits

$$\xi^k(t) = (b_k - a_k \text{sn}^2 \delta_k t, -2\delta_k a_k \text{sn} \delta_k t \text{ cn} \delta_k t \text{ dn} \delta_k t)$$

(16)
on the $\xi$-plane for $k \in (0, 1)$, where sn, cn and dn are the Jacobi elliptic functions, $k$ is the elliptic modulus and

$$a_k = \frac{3k^2}{2\sqrt{k^4 - k^2 + 1}}, \quad b_k = \frac{k^2 + 1}{2\sqrt{k^4 - k^2 + 1}} - \frac{1}{2}, \quad \delta_k = \frac{1}{2\sqrt{k^4 - k^2 + 1}}.$$  

See Fig. 2 for these periodic orbits on the $\xi$-plane. In particular, there is a center at $\xi = 0$, from which another family of periodic orbits not lying on the $\xi$-plane bifurcate if $1/\omega \notin \mathbb{Z}$ at least (see Appendix A). Pitchfork and period-doubling bifurcations, which correspond to $n = 1$ in our theory (cf. Remark 4.2), of the family of periodic orbits in (14) with $c_1 = -d \neq 0$ and $c_2 = 0$ were studied in [2, 3].

The period and (Hamiltonian) energy of the periodic orbit, $T_k$ and $H_k = H(\xi^k(t), 0)$, are given by

$$T_k = \frac{2K(k)}{\delta_k} = 4K(k)\sqrt{k^4 - k^2 + 1},$$

$$H_k = \frac{(2 - k^2)(2k^2 - 1)(k^2 + 1)}{24(k^4 - k^2 + 1)^{3/2}} + \frac{1}{12},$$

where $K(k)$ is the complete elliptic integral of the first kind. Both of $T_k$ and $H_k$ increase monotonically in $k$ with $\lim_{k \to 0} T_k = 2\pi$, $\lim_{k \to 1} T_k = \infty$, $\lim_{k \to 0} H_k = 0$ and $\lim_{k \to 1} H_k = \frac{1}{6}$. See, e.g., [4, 23] for necessary information on the elliptic functions and elliptic integrals. Thus, assumptions (A1)-(A3) hold with the parameter $\alpha = k$. We also see that $\text{Fix}(R) = \{\xi_2, \eta_2 = 0\}$ and $\text{Fix}(-R) = \{\xi_1, \eta_1 = 0\}$, and take $\tilde{\zeta} = (1, 0)$ and $\zeta = (0, 1)$. The NVE (12) becomes

$$\dot{v}_1 = v_2, \quad \dot{v}_2 = -(\omega^2 + 2c_1 \xi^k(t))v_1.$$  

(17)

In the following, we apply our theory to (14) separately for $c_1 = 0$ and $c_1 \neq 0$.  

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{One parameter family of symmetric periodic orbits $\xi^k(t)$ in (14).}
\end{figure}
4.1. Case of $c_1 = 0$

When $c_1 = 0$, the fundamental matrix of (17) is given by

$$
\Psi^k(t) = \begin{pmatrix}
\cos \omega t & \frac{1}{\omega} \sin \omega t \\
-\omega \sin \omega t & \cos \omega t
\end{pmatrix}.
$$

The Melnikov function (13) becomes

$$
M^n(k) = -\omega \sin n\omega T_k.
$$

We see that $M^n(k)$ has a simple zero if and only if $n\omega T_k = m\pi$, i.e.,

$$
m\Omega_k = 2n\omega
$$

for some $m \in \mathbb{N}$ relatively prime to $n$, where

$$
\Omega_k = \frac{2\pi}{T_k} = \frac{\pi}{2K(k) \sqrt{k^4 - k^2 + 1}}.
$$

Moreover,

$$
\mu_k^n = \begin{pmatrix}
\cos m\pi \\
0
\end{pmatrix}
$$

when condition (18) holds. Applying Theorem 3.6, we obtain the following result.

**Theorem 4.1.** Let $c_1 = 0$. Suppose that condition (18) holds at $k = k_0$ for $m$ even (resp. odd) and relatively prime to $n$. Then two branches (resp. one branch) of symmetric periodic orbits appear (resp. appears) at $k = k_0$, and the period of the periodic orbits tend to $nT_{k_0}$ (resp. to $2nT_{k_0}$) as $k \to k_0$. If condition (18) does not hold at $k = k_0$ for any $m, n \in \mathbb{N}$, then such a family of symmetric periodic orbits does not exist.

**Remark 4.2.** The bifurcations detected by Theorem 4.1 for $n = 1$ correspond to standard pitchfork and period doubling bifurcations when $m$ is even and odd, respectively. Note that $m$ must be odd when condition (18) holds for $n$ even.
Above all, the family (16) undergoes infinitely many bifurcations and the bifurcation points are dense in the set of $H_k$ for each value of $\omega$. Theorem 3.4 also implies that the infinitely many families may exhibit infinitely many bifurcations and very complicated structures of symmetric periodic orbits exist in (14). It is natural to consider this behavior to be typical in reversible systems. In Fig. 3, bifurcation curves for $n \leq 5$ in (14) are plotted in the $(H_k, \omega)$-plane.

We give numerical computation results for (14) with $c_1 = 0$ in Figs. 4 and 5, respectively, which show bifurcation curves for $n \leq 3$ and periodic orbits for $H = 0.2$ when $c_2 = \frac{1}{8}$ and $\omega = \omega_*$. Another family of symmetric periodic orbits born at the origin (see Appendix A) is drawn in black in Fig. 4. We see that many bifurcations occur and several types of symmetric periodic orbits are born even for $n \leq 3$. To obtain these results, we considered the boundary value problem for (14) with boundary conditions

$$\xi_2(0) = \eta_2(0) = \xi_2(\tilde{T}) = \eta_2(\tilde{T}) = 0,$$

which mean that $(\xi(t), \eta(t)) \in \text{Fix}(R)$ at $t = 0, \tilde{T}$, where $\tilde{T} = T$ or $T/2$. By the reversibility, the solution $(\xi(t), \eta(t))$ gives an $R$-symmetric periodic orbit of (14). We solved the boundary value problem and continued the solution with $\tilde{T}$ to obtain a one-parameter family of $R$-symmetric periodic orbits. The numerical continuation tool AUT097 [8] was used to perform this computation, and the equilibrium $(\xi, \eta) = (0, 0)$ was chosen as the starting solution. Here the boundary values $\xi_1(0), \xi_1(\tilde{T}), \eta_1(0)$ and $\eta_1(\tilde{T})$ were also taken as free parameters and the Hamiltonian energy $H$ was monitored. A similar numerical approach was also used for computing symmetric relative periodic orbits in the isosceles three-body problem [19].

4.2. Case of $c_1 \neq 0$

4.2.1. General case When $c_1 \neq 0$, it is difficult to obtain an analytical expression for the fundamental matrix of (17) in general. However, we apply classical results on Hill’s
Figure 5. Numerically computed symmetric periodic orbits bifurcating from the family (16) in (14) for $c_1 = 0$, $c_2 = \frac{1}{6}$, $\omega = \omega_*$ and $H = 0.2$: (a) $(m, n) = (1, 4)$; (b) $(1, 5)$; (c) $(1, 6)$; (d) $(1, 7)$; (e) $(2, 7)$; (f) $(2, 9)$; (g) $(3, 10)$; (h) $(3, 11)$.

equations [13] to obtain general results.

We first change the independent variable $t \mapsto \delta k t$ to rewrite (17) as the Lamé equation

$$\ddot{\chi} + (k^2 - \ell(\ell + 1)k^2\sin^2 t)\chi = 0,$$

(19)
where \( \chi = v_1 \) and
\[
\begin{align*}
\h_k &= \frac{\omega^2 + 2c_1 b_k}{\delta_k^2} = 4 \left( (k^2 + 1)c_1 + (\omega^2 - c_1)\sqrt{k^4 - k^2 + 1} \right), \\
\ell(\ell + 1) &= \frac{2c_1 a_k}{k^2 \delta_k^2} = 12c_1.
\end{align*}
\]

Even when \( k \) changes, \( \ell \) does not change if \( c_1 \) is fixed. Note that \( \text{sn}^2 t \) has a period of \( 2K(k) \) since \( \text{sn} t \) is \( 4K(k) \)-periodic.

Let \( c_1 \) (i.e., \( \ell \)) and \( \omega \) be fixed and let \( k \) be varied. Define the discriminant for (19) as
\[
\Delta(k) = \chi_1(2K(k); k) + \chi_2(2K(k); k),
\]
where \( \chi_j(t; k), j = 1, 2, \) are solutions of (19) satisfying
\[
\chi_1(0; k) = 1, \quad \dot{\chi}_1(0; k) = 0, \quad \chi_2(0; k) = 0, \quad \dot{\chi}_2(0; k) = 1.
\]

Suppose that the characteristic equation
\[
\rho^2 - \Delta(k) \rho + 1 = 0
\]
has different roots (characteristic multipliers) \( \rho_j, j = 1, 2. \) Then the monodromy matrix \( \mu_k \) of (17) is expressed as
\[
\mu_k = Q^{-1} \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix},
\]
where \( Q \) is a \( 2 \times 2 \) nonsingular matrix, since that of (19) has the same form by the Floquet Theorem [13]. Moreover, when \( k > 0 \) is sufficiently small, the characteristic multipliers of (19) are approximated by those of the Mathieu equation
\[
\ddot{\chi} + (h_k + 6c_1 k^2 \cos 2t)\chi = 0.
\]
It follows from a well-known formula [1] that when the characteristic multipliers of (21) are denoted by \( e^{\pm i \pi \lambda_k} \), we have
\[
\h_k = \lambda_k^2 + \frac{9c_1^2 k^4}{2(\lambda_k^2 - 1)} + O(k^8)
\]
for \( k \approx 0 \) if \( \lambda_k \notin \mathbb{N} \). From these observations we can easily obtain the following result.

**Theorem 4.3.** When \( 2\omega \notin \mathbb{N} \), there exist infinitely many families of symmetric periodic orbits bifurcating from the family (16) in (14).

**Proof.** Assume that \( 2\omega \notin \mathbb{N} \). From the estimate (22) we can write \( \rho_1 = e^{i\pi \lambda_k} \) and \( \rho_2 = e^{-i\pi \lambda_k} \) for \( k > 0 \) sufficiently small, where \( \lambda_k \in \mathbb{R}\setminus\mathbb{N} \) is not a constant function of \( k \). On the other hand, the function \( \Delta(k) \) is analytic in \( k \) since \( \chi_j(t; k), j = 1, 2, \) are so in \( t \) and \( k \). Hence, \( \lambda_k \) also depends on \( k \) analytically since \( e^{\pm i \pi \lambda_k} \) are roots of the characteristic equation (20). Thus, we see that
\[
\lambda_k = \frac{m}{n}
\]
has a simple zero \( k = k_{m/n} \) for infinitely many pairs \( (m, n) \), where \( m, n \in \mathbb{N} \) are relatively prime with \( n \neq 1 \).

On the other hand, we compute the Melnikov function as
\[
M^n(k) = -\frac{2i q_{11} q_{21}}{\det Q} \sin n \pi \lambda_k,
\]
where \( q_{ij} \) is the \((i, j)\)-element of \( Q \). Hence, \( M^n(k) \) has a simple zero at \( k = k_{m/n} \) if Eq. (23) does so. Thus, we apply Theorem 3.6 to obtain the result.

4.2.2. Case of \( c_1 \approx 0 \) For \( c_1 \approx 0 \) we can compute the monodromy matrix \( \mu_k \) of (17) approximately and estimate bifurcation points using Theorem 3.6 as we see below. We introduce a small parameter, denoted by \( \varepsilon \) again, such that \( 0 < \varepsilon \ll 1 \), and set \( c_1 = \varepsilon c_1 \).

Suppose that \( m \Omega_k \approx 2n \omega \) and set
\[
\varepsilon \nu_k = \frac{m^2 \Omega_k^2 - 4n^2 \omega^2}{4n^2}.
\]
Using the van der Pol transformation
\[
\begin{pmatrix}
\rho_1 \\
\rho_2
\end{pmatrix} = \begin{pmatrix}
\cos \frac{m \Omega_k}{2n} t - \frac{2n}{m \Omega_k} \sin \frac{m \Omega_k}{2n} t \\
- \sin \frac{m \Omega_k}{2n} t - \frac{2n}{m \Omega_k} \cos \frac{m \Omega_k}{2n} t
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]
in (17), we have
\[
\begin{pmatrix}
\dot{\rho}_1 \\
\dot{\rho}_2
\end{pmatrix} = -\frac{\varepsilon n}{m \Omega_k} (\nu_k - 2\bar{c}_1 \varepsilon_1^k(t))
\times \left(\rho_1 \cos \frac{m}{n} \Omega_k t - \rho_2 \sin \frac{m}{n} \Omega_k t\right)
\left(\sin \frac{m}{n} \Omega_k t\right).
\]
Applying the averaging method (see, e.g., [9, 16]) to (26), we obtain
\[
\begin{align*}
\dot{\rho}_1 &= \frac{\varepsilon n}{2m \Omega_k} \left[\nu_k - \bar{c}_1 \left(2\beta_0(k) - \gamma_{m/n}(k)\right)\right] \rho_2, \\
\dot{\rho}_2 &= -\frac{\varepsilon n}{2m \Omega_k} \left[\nu_k - \bar{c}_1 \left(2\beta_0(k) + \gamma_{m/n}(k)\right)\right] \rho_1, \\
\end{align*}
\]
where
\[
\begin{align*}
\beta_0(k) &= \frac{k^2 + 1}{2 \sqrt{k^4 - k^2 + 1}} - \frac{3(K(k) - E(k))}{2K(k) \sqrt{k^4 - k^2 + 1}} - \frac{1}{2}, \\
\beta_\ell(k) &= \frac{3\ell \pi^2}{2K^2(k) \sqrt{k^4 - k^2 + 1}} \cosech \left(\frac{\ell \pi K(k')}{K(k)}\right), \quad \ell \in \mathbb{N},
\end{align*}
\]
and
\[
\gamma_{m/n}(k) = \begin{cases}
\beta_{2m}(k) & \text{for } n = 1; \\
\beta_m(k) & \text{for } n = 2; \\
0 & \text{for } n > 2.
\end{cases}
\]
Here $E(k)$ is the complete elliptic integral of the second kind and $k'$ is the complementary elliptic modulus $k'^2 = 1 - k^2$. We have also used the relation
\[
\xi_k^1(t) = b_k - a_k \left[ \frac{K(k) - E(k)}{k^2 K(k)} - \frac{\pi^2}{k^2 K^2(k)} \sum_{\ell=1}^{\infty} \ell \cosech \left( \frac{\ell \pi K(k')}{K(k)} \right) \cos \ell \Omega_k t \right]
= \sum_{\ell=0}^{\infty} \beta_\ell(k) \cos \ell \Omega_k t,
\]
to obtain the averaged system (27). It follows from the averaging theorem that solutions of (27) approximate those of (26) up to $O(\varepsilon)$ on a time interval of $O(1)$ (not of $O(1/\varepsilon)$).

We can regard the fundamental matrix of the averaged system (27) as a function of $\varepsilon t$, and write it as $\tilde{\Psi}^k(\varepsilon t)$. Let
\[
\nu_+^k = \frac{n}{2m\Omega_k} \left[ \nu_k - c \left( 2\beta_0(k) \pm \gamma_{m/n}(k) \right) \right].
\]
Note that $\nu_+^k > \nu_-^k$ for $n = 1, 2$ since $\beta_m(k) > 0$ for any $m \in \mathbb{N}$, and that $\nu_-^k = \nu_+^k$ for $n > 2$. Assuming that $\tilde{\Psi}^k(0) = \text{id}_2$, we have
\[
\tilde{\Psi}^k(\tau) = \begin{pmatrix} \cos \sqrt{\nu_+^k \nu_-^k} \tau & \pm \sqrt{\frac{\nu_-^k}{\nu_+^k}} \sin \sqrt{\nu_+^k \nu_-^k} \tau \\ \mp \sqrt{\frac{\nu_-^k}{\nu_+^k}} \sin \sqrt{\nu_+^k \nu_-^k} \tau & \cos \sqrt{\nu_+^k \nu_-^k} \tau \end{pmatrix}
\]
if $\nu_+ \nu_- > 0$, where the upper and lower signs are taken for $\nu_+ > 0$ and $\nu_- < 0$, respectively, and
\[
\tilde{\Psi}^k(\tau) = \begin{pmatrix} \cosh -\sqrt{\nu_+^k \nu_-^k} \tau & -\sqrt{\frac{\nu_-^k}{\nu_+^k}} \sinh \sqrt{\nu_+^k \nu_-^k} \tau \\ \sqrt{\frac{\nu_-^k}{\nu_+^k}} \sinh \sqrt{\nu_+^k \nu_-^k} \tau & \cosh -\sqrt{\nu_+^k \nu_-^k} \tau \end{pmatrix}
\]
if $\nu_+ \nu_- < 0$. Hence, the fundamental matrix $\Psi^k(t)$ of (17) is given by
\[
\Psi^k(t) = \left( \begin{array}{cc} \cos \frac{m\Omega_k t}{2n} & -2n \sin \frac{m\Omega_k t}{2n} \\ -\sin \frac{m\Omega_k t}{2n} & -2n \cos \frac{m\Omega_k t}{2n} \end{array} \right)^{-1} \tilde{\Psi}^k(\varepsilon t) \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{2n}{m\Omega_k} \end{array} \right) + O(\varepsilon^2),
\]
so that its monodromy matrix is written as
\[
\mu_k^n = \left( \begin{array}{cc} \cos \pi m & 0 \\ 0 & -\frac{2n}{m\Omega_k} \cos \pi m \end{array} \right)^{-1} \tilde{\Psi}^k(\varepsilon nT_k) \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{2n}{m\Omega_k} \end{array} \right) + O(\varepsilon^2).
Thus we compute
\[
M^n(k) = \begin{cases} 
\frac{m\Omega_k}{2n} \sqrt{\frac{\nu_+^k \nu_-^k}{\nu_+^k}} \sin \varepsilon nT_k \sqrt{\nu_+^k \nu_-^k} + O(\varepsilon^2) & \text{for } \nu_+^k, \nu_-^k > 0; \\
-\frac{m\Omega_k}{2n} \sqrt{-\frac{\nu_+^k \nu_-^k}{\nu_+^k}} \sinh \varepsilon nT_k \sqrt{-\nu_+^k \nu_-^k} + O(\varepsilon^2) & \text{for } \nu_+^k < 0 < \nu_-^k; \\
-\frac{m\Omega_k}{2n} \sqrt{\frac{\nu_+^k \nu_-^k}{\nu_-^k}} \sin \varepsilon nT_k \sqrt{\nu_+^k \nu_-^k} + O(\varepsilon^2) & \text{for } \nu_+^k, \nu_-^k < 0,
\end{cases}
\]
(28)

and
\[
\mu_k^\nu \hat{\zeta} = \begin{cases} 
\left( \pm \cos \varepsilon nT_k \sqrt{\nu_+^k \nu_-^k} + O(\varepsilon^2) \right) & \text{for } \nu_+^k \nu_-^k > 0; \\
0 & \text{for } \nu_+^k < 0 < \nu_-^k
\end{cases}
\]
(29)

if \(M^n(k) = 0\), where the upper and lower signs are taken for \(m\) even and odd, respectively. Note that
\[
M^n(k) \to \varepsilon m \pi \nu_+^k + O(\varepsilon^2) < 0
\]
as \(\nu_-^k \to 0\) if \(\nu_+^k \neq \nu_-^k\). Since \(M^n(k)\) and \(\mu_k^\nu \hat{\zeta}\) are smooth with respect to \(k\) by the smoothness of solutions on parameters, we easily obtain the following result via Theorem 3.6.

**Theorem 4.4.** Let \(c_1 = \varepsilon \bar{c}_1 \neq 0\), where \(0 < \varepsilon \ll 1\). Let \(m, n\) be relatively prime integers and let \(m\) be even (resp. odd). Suppose that \(m\Omega_k \approx 2n\omega\), and define the parameter \(\nu_k\) by (24). Then two branches (resp. one branch) of symmetric periodic orbits appear (resp. appears) at some \(k = k_0\) such that
\[
\nu_k = c_1(2\beta_0(k) + \gamma_{m/n}(k)) + O(\varepsilon).
\]
Moreover, the period of the periodic orbits tend to \(nT_{k_0}\) (resp. to \(2nT_{k_0}\)) as \(k \to k_0\).

**Proof.** Applying the implicit function theorem to (28) we see that \(M^n(k)\) has a simple zero near \(\nu_+ = 0\). Moreover, it follows from (29) that if \(\nu_+ = 0\), then \(\mu_k^\nu \hat{\zeta} = \hat{\zeta} + O(\varepsilon^2)\) for \(m\) even and \(\mu_k^\nu \hat{\zeta} = -\hat{\zeta} + O(\varepsilon^2)\) for \(m\) odd. By Theorem 3.6 we obtain the result.

**Remark 4.5.** Recently, Maier [12] gave an efficient method to compute the monodromy matrix of the Lamé equation (19) for \(\ell \in \mathbb{Z}\). His method can be used for necessary computations when Theorem 3.6 is applied to the case of \(\ell \in \mathbb{Z}\) in (14).

**Appendix A. Another family of symmetric periodic orbits in (14)**

As in Theorem 1 of [19], we use a classical result of Devaney [7] to obtain the following.

**Proposition A. 1.** Another family of symmetric periodic orbits bifurcates from the origin in (14) if \(1/\omega \not\in \mathbb{Z}\).
Proof. The center at the origin has two pairs of purely imaginary eigenvalues, \( \pm i \) and \( \pm i\omega \). Hence, the result immediately follows from Theorem 8.1 of [7].

Remark A. 2. When \( 1/\omega \in \mathbb{Z} \), we can appeal to the results in Chapter 6 of Sevryuk [17]. Under some nondegeneracy conditions, two family of symmetric periodic orbits are also born from the center and interact so that several types of bifurcations occur. See [17] for more details.

Acknowledgments

This work was partially supported by the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) Nos. 21540124 and 22540180. The author thanks Mitsuru Shibayama for helpful discussions.

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