HIGHER-ORDER MELNIKOV METHOD AND CHAOS FOR TWO-DEGREE-OF-FREEDOM HAMILTONIAN SYSTEMS WITH SADDLE-CENTERS

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Abstract. We consider two-degree-of-freedom Hamiltonian systems with saddle-centers, and develop a Melnikov-type technique for detecting creation of transverse homoclinic orbits by higher-order terms. We apply the technique to the generalized Hénon-Heiles system and give a positive answer to a remaining question of whether chaotic dynamics occurs for some parameter values although it is known to be nonintegrable in a complex analytical meaning.

1. Introduction. In this paper we consider two-degree-of-freedom Hamiltonian systems of the form

\[ \dot{x} = J \mathcal{D}_x H(x, y), \quad \dot{y} = J \mathcal{D}_y H(x, y), \quad (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad (1.1) \]

where the dot represents differentiation with respect to time \( t \), \( H : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is \( C^{r+1} \) (\( r \geq 4 \)) and \( J \) is the 2 \( \times \) 2 symplectic matrix,

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

We especially assume that the \( x \)-plane is invariant under the flow of (1.1) and there is a saddle-center at the origin \((x, y) = (0, 0) (= O)\) with a homoclinic orbit on the \( x \)-plane. Here a “saddle-center” is an equilibrium at which the Jacobian matrix has a pair of positive and negative real eigenvalues and a pair of purely imaginary eigenvalues. See Sect. 2 for our precise assumption. In this situation, via the Liapunov center theorem [1, 21], there exist a one-parameter family of periodic orbits near the saddle-center.

Complicated dynamics of two-degree-of-freedom Hamiltonian systems with saddle-centers have been studied by several researchers [10, 11, 18, 19, 22, 33]. In the earlier work of [19, 22] a Shil’nikov-type approach [28, 30] was used for general “analytic” Hamiltonian systems to show the presence of horseshoes, which also implies the occurrence of chaos, near the saddle-centers under a degenerate condition. For a more restricted class of systems with potentials including (1.3) below, the non-degenerate condition was represented in a convenient form which is independent of the coordinates in [10]. Countable infinities of multi-pulse homoclinic orbits and periodic orbits in perturbed systems or near the homoclinic loops were also discussed in [11, 18, 22].

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In [33] a Melnikov type technique was developed for detecting the existence of orbits transversely homoclinic to periodic orbits near the saddle-center and hence the occurrence of chaos in (1.1).

To apply this technique, we only compute a function called the Melnikov function and determine whether it has a simple zero or not (see Sect. 3, especially Theorem 3.1, for the details), like the standard Melnikov method [12, 31]. The obtained condition for chaos is equivalent to the previous one of [19, 22] in analytic Hamiltonian systems. This type of approach is often superior to the Shil’nikov-type at the point that it is extensible to higher-degree-of-freedom systems and higher-order approximations as in [4, 14, 20, 30]. So the technique of [33] was actually extended to three or more degrees of freedom systems in [37] for studying homoclinic or heteroclinic connections between whiskered invariant tori. The presence of such heteroclinic connections implies the occurrence of Arnold diffusion type motions (see [37] for the details), a numerical evidence of which was given for a three-degree-of-freedom system in [35]. These techniques were applied to an infinite-degree-of-freedom system representing a mathematical model of an undamped, buckled beam in [34]. Equivalence of the criterion for chaos [10, 33] and nonintegrability in a complex analytic meaning [23, 25] was also discussed in [23, 24, 36].

Our special attention is paid to the (generalized) Hénon-Heiles Hamiltonian

\[ H(x, y) = \frac{1}{2} (x_1^2 + x_2^2 + y_1^2 + y_2^2) + cx_1y_1^2 + \frac{1}{3} dx_1^3 \]  

(1.2)

with \( c/d = 3/4 \). The original system studied by Hénon and Heiles [15] is the case of \( c = 1 \) and \( d = -1 \) in (1.2). The Hamiltonian (1.2) has a saddle-center at \((x_1, x_2, y_1, y_2) = (-1/2c, 0, \pm \sqrt{2 - (d/c)/2c}, 0)\) if \( c/d > 1/2 \), and at \((-1/d, 0, 0, 0)\) if \( c/d < 1/2 \). When \( c/d > 1/2 \) (resp. \( c/d < 1/2 \)), it is transformed to

\[ H(x, y) = \frac{1}{2} (x_2^2 + y_2^2) + \frac{1}{2} (-x_1^2 + \omega^2 y_1^2) + \frac{1}{3} \beta_1 x_1^3 + \frac{\beta_2}{2} x_1 y_1^2 + \frac{\beta_3}{3} y_1^3 \]  

(1.3)

by a change of coordinates

\[ \tilde{x}_1 = \frac{x_1 - x_{10} + \mu y_1}{\sqrt{1 + \mu^2}}, \quad \tilde{x}_2 = \frac{x_2 + \mu y_2}{\sqrt{1 + \mu^2}}, \]

\[ \tilde{y}_1 = \frac{-\mu(x_1 - x_{10}) + y_1}{\sqrt{1 + \mu^2}}, \quad \tilde{y}_2 = \frac{-\mu x_2 + y_2}{\sqrt{1 + \mu^2}} \]

(resp. \( \tilde{x}_1 = x_1 - 1/d, \tilde{x}_2 = x_2, \tilde{y}_1 = y_1, \tilde{y}_2 = y_2 \)), where the tilde represents the old coordinates and

\[ \beta_1 = \frac{2c}{\sqrt{1 + \mu^2}}, \quad \beta_2 = \frac{2(d - c)}{\sqrt{1 + \mu^2}}, \quad \beta_3 = \frac{\mu(c + d)}{\sqrt{1 + \mu^2}}, \]

\[ \omega = \frac{\mu}{\sqrt{2}}, \quad x_{10} = \frac{1 + \mu^2}{2c}, \quad \mu = \sqrt{2 - \frac{d}{c}} \]  

(1.4)

\[ \text{resp.,} \quad \beta_1 = d, \quad \beta_2 = 2c, \quad \beta_3 = 0, \quad \omega = \sqrt{1 - \frac{2c}{d}} \].
In [10, 33] it was shown that there exist horseshoes near the saddle-centers and chaotic dynamics occurs in (1.1) with (1.3) if
\[ \frac{\beta_2}{\beta_1} \neq \frac{l(l + 1)}{6} \] for any non-negative integer \( l \), which corresponds to
\[ \frac{c}{d} \neq 0, \frac{1}{6}, \frac{1}{2}, \frac{3}{4}, 1 \] in (1.2). On the other hand, using Ziglin's method [40], Ito [16, 17] showed that the Hamiltonian (1.2) is nonintegrable in a complex analytic meaning if
\[ \frac{c}{d} \neq 0, \frac{1}{6}, \frac{1}{2}, \frac{1}{2} \] in (1.5). It is actually integrable in the three cases except \( \frac{c}{d} = 1/2 \) (see, e.g., [9]) and it was recently proven to be nonintegrable in the complex analytic meaning when \( \frac{c}{d} = 1/2 \) (see [23]). So we have a question of whether transverse homoclinic orbits exist and chaotic dynamics occurs when \( \frac{c}{d} = 1/2 \), 3/4 although the Melnikov function is identically zero and the Melnikov technique of [33] does not apply but it is nonintegrable in the complex analytic meaning.

The object of this paper is to extend the idea of [33] to second-order approximations and develop a technique for detecting the existence of transverse homoclinic orbits even when the techniques of [10, 33] are not applicable. So we give a positive answer to the above question on the Hénon-Heiles Hamiltonian (1.2) with \( \frac{c}{d} = 3/4 \). The basic ideas used here are similar to those in [33], which the reader should consult for the details and proofs of some preliminary but key results. The outline of this paper is as follows. In Sect. 2 we give our precise assumptions on (1.1) and follow [33] to describe their immediate consequences on the phase space structure. We state our main result after giving the main result of [33] in Sect. 3 and provide its proof in Sect. 4. In Sect. 5 we illustrate our theory for the Hamiltonian (1.3) with \( \frac{\beta_2}{\beta_1} = 1/3 \) including the Hénon-Heiles Hamiltonian (1.2) with \( \frac{c}{d} = 3/4 \) as a special case. We remark that for \( \frac{c}{d} = 1/2 \) our technique does not directly apply to (1.2) since the equilibrium with a homoclinic orbit is a saddle-parabolic type for which the Jacobian matrix has a double zero eigenvalue. An extension of our result to that case, in which a higher-order approximation is also required will be reported elsewhere.

2. Assumptions and the phase space. We make the following assumptions on (1.1).

(A1) \( D_x H(0, 0) = 0 \) and \( D_y H(x, 0) = 0 \) for any \( x \in \mathbb{R}^2 \).

Assumption (A1) means that the origin \( O \) is an equilibrium and the \( x \)-plane, \( \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 | y = 0\} \), is invariant under the flow of (1.1). The system restricted on the \( x \)-plane,
\[ \dot{x} = JD_x H(x, 0), \]
has an equilibrium at \( x = 0 \).

(A2) The matrix \( JD_x^2 H(0, 0) \) has a pair of real eigenvalues \( \pm \lambda (\lambda > 0) \) so that the equilibrium \( x = 0 \) of (2.1) is a hyperbolic saddle. Moreover, there is a homoclinic orbit \( x^h(t) \) to the saddle \( x = 0 \). See Fig. 1. Let \( \Gamma_0 = \{ x^h(t) | t \in \mathbb{R} \} \cup \{ 0 \} \).
(A3) The matrix $J D_y^2 H(0,0)$ has a pair of purely imaginary eigenvalues $\pm i \omega$ ($\neq 0$).

Let
\[ p_j(x,\eta) = \frac{1}{(j+1)!} D^{j+1} f(x,0) (\eta, \ldots, \eta), \quad j = 0, 1, \ldots, \]
with $f(x,y) = JD_y H(x,y)$, where $\eta \in \mathbb{R}^2$. Note that $p_0(x,\eta) \equiv 0$ by assumption (A1).

(A4) $p_1(x,\eta) \neq 0$.

The Hamiltonian (1.3) satisfies assumptions (A1)-(A4) if $\beta_1, \beta_2 \neq 0$. In particular, the homoclinic orbit $x^h(t)$ is given by
\[ x^h(t) = \left( \frac{3}{2\beta_1} \text{sech}^2 \left( \frac{t}{2} \right), -\frac{3}{2\beta_1} \text{sech}^2 \left( \frac{t}{2} \right) \tanh \left( \frac{t}{2} \right) \right). \quad (2.2) \]

Assumptions (A2) and (A3) mean that the equilibrium $O$ is a saddle-center and has a homoclinic orbit $(x,y) = (x^h(t),0)$. It follows from the center manifold theory (e.g., [12,29]) that the saddle-center $O$ has a $C^\infty$, two-dimensional local center manifold, $W^c_{\text{loc}}(O)$, which may be non-unique, as well as $C^\infty$, one-dimensional stable and unstable manifolds, $W^s(O)$ and $W^u(O)$, which coincide along the homoclinic orbit $(x^h(t),0)$. The case in which assumption (A4) does not hold was also studied in [33].

Now we describe immediate consequences of the above assumptions on the phase space structure of (1.1). These results were proven in [33] and also play an important role below.

Using the Liapunov center theorem [1, 21], we obtain the following result (see Proposition 2.1 of [33]).

**Proposition 2.1.** There exists a one-parameter family of periodic orbits,
\[ \gamma^\alpha, \quad 0 < \alpha \leq \alpha_0, \]
that approach $O$ as $\alpha \to 0$ and whose periods approach $2\pi/\omega$. Moreover, for $\alpha_0 > 0$ sufficiently small the periodic orbits fill up a $C^r$, two-dimensional, normally hyperbolic invariant manifold with boundary,

$$\mathcal{M} = \bigcup_{\alpha = 0}^{\alpha_0} \gamma^\alpha \subset W_{\text{loc}}^c(O),$$

which is tangent to the $y$-plane, $\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 | x = 0\}$, at $O$.

We denote $H_0 = H(\gamma^0)$. Without a loss of generality we can assume that $H_0 = H(0, 0) = 0$.

Let $U_{\epsilon_1, \epsilon_2} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 | |x| < \epsilon_1, |y| < \epsilon_2\}$ be a neighborhood of the saddle-center $O$ including the invariant manifold with boundary $\mathcal{M}$ in $\mathbb{R}^2 \times \mathbb{R}^2$. A different neighborhood was used in [33] but both are essentially the same. Denote by $\phi_t$ the flow generated by (1.1). From the invariant manifold theory of Fenichel [8] (see also [32]) we have the following result (see Proposition 2.3 of [33] and Proposition 1 of [37]).

**Proposition 2.2.** For $\epsilon_j > 0$, $j = 1, 2$, sufficiently small, there exist $C^r$, three-dimensional, locally invariant manifolds in $U_{\epsilon_1, \epsilon_2}$, denoting by $W^s_{\text{loc}}(\mathcal{M})$ and $W^u_{\text{loc}}(\mathcal{M})$, having the following properties.

(i) $W^s_{\text{loc}}(\mathcal{M}) \cap W^u_{\text{loc}}(\mathcal{M}) = \mathcal{M}$,

(ii) Let $z^{s,u}(t) = (x^{s,u}(t), y^{s,u}(t))$ be a trajectory starting in $W^s_{\text{loc}}(\mathcal{M})$ at $t = 0$. Then as $t \to +\infty$ (resp. $t \to -\infty$), either

(iia) $x^{s}(t)$ (resp. $x^{u}(t)$) crosses $\partial U_{\rho,0}$;

or

(iib) $z^{s}(t) \to \mathcal{M}$ (resp. $z^{u}(t) \to \mathcal{M}$).

Moreover, $W^s_{\text{loc}}(\mathcal{M})$ can be represented as

$$W^s_{\text{loc}}(\mathcal{M}) = \{ (x, y) \in U_{\epsilon_1, \epsilon_2} | x \in \Gamma_y \},$$

where $\Gamma_y \subset \mathbb{R}^2$ is $\mathcal{O}(|y|^2)$-close to $\Gamma_0$ in $U_{\epsilon_1, \epsilon_2}$.

We refer to $W^s_{\text{loc}}(\mathcal{M})$ and $W^u_{\text{loc}}(\mathcal{M})$ as the local stable and unstable manifolds of $\mathcal{M}$. We also have the following result (see Proposition 2.4 of [33].)

**Proposition 2.3.** Let $\gamma^\alpha_0(t)$ be orbits on $W^s_{\text{loc}}(\mathcal{M})$ such that $H(\gamma^\alpha_0(t)) = H_0$. Then for $\alpha > 0$ sufficiently small there exists a periodic orbit $\gamma^\alpha \in \mathcal{M}$ such that $\text{dist}(\gamma^\alpha(t), \gamma^\alpha_0(t)) \to 0$ (resp. $\text{dist}(\gamma^\alpha(t), \gamma^\alpha_0(t)) \to 0$) as $t \to +\infty$ (resp. $t \to -\infty$), where $\text{dist}(x, A) = \inf_{z \in A} |x - z|$ for $A \subset \mathbb{R}^4$. This implies that $\gamma^\alpha_0(t) \in W^s(\gamma^\alpha)$ and $\gamma^\alpha_0(t) \in W^u(\gamma^\alpha)$, where $W^s(\gamma^\alpha)$ and $W^u(\gamma^\alpha)$, respectively, denote the stable and unstable manifolds of $\gamma^\alpha$ in a usual meaning.

3. Main result. Consider the variational equations of (1.1) in the $y$-direction about the saddle-center $O$ and homoclinic orbit $(x, y) = (x^h(t), 0)$,

$$\dot{\eta} = JD^2_p H(0, 0) \eta$$

(3.1)

and

$$\dot{\eta} = JD^2_p H_0(x^h(t), 0) \eta.$$  

(3.2)

We call (3.1) and (3.2) the normal variational equations (NVEs) of (1.1) about the saddle-center $O$ and the homoclinic orbit $(x, y) = (x^h(t), 0)$, respectively. Let $\Phi(t)$ and $\Psi(t)$ be fundamental matrices to (3.1) and (3.2), respectively, such that
Φ(0) = Ψ(0) = id, where id is the 2 × 2 identity matrix. In particular, all elements of Φ(t) can be written as linear combinations of sin ωt and cos ωt (see [36, 37]). We can easily show that the limits

\[ B_\pm = \lim_{t \to \pm \infty} \Phi(-t) \Psi(t) \]

exist and \( B_\pm \) are nonsingular matrices (see Lemma 3.1 of [33]). We set \( B_0 = B_+ B_-^{-1} \). Let

\[ q_j(x, \eta) = \frac{1}{(j+2)!} D_{H}^{j+2} H(x,0)(\eta, \ldots, \eta), \quad j = 0, 1, \ldots \]

where \( \eta \in \mathbb{R}^2 \). Note that \( p_{j+1}(x, \eta) = JD_x q_j(x, \eta), \quad j = 0, 1, \ldots \) In particular, we have

\[ q_0(x, \eta) = \frac{1}{2} \eta^T D_x^2 H(x,0) \eta, \]

where \( T \) is the transpose operator.

Let \( e_1 = (1, 0)^T \in \mathbb{R}^2 \). Define the first-order Melnikov function as

\[ M_1(t_0) = q_0(0, e_1) - q_0(0, B_0 \Phi(t_0)e_1). \]

The following result was proven in [33].

**Theorem 3.1.** Suppose that \( M_1(t_0) \) has a simple zero, i.e.,

\[ M_1(t_0) = 0, \quad \frac{d}{dt} M_1(t_0) \neq 0 \]

for some \( t_0 \in \mathbb{R} \). Then for \( \alpha > 0 \) sufficiently small there exist orbits transversely homoclinic to \( \gamma_\alpha \) and the Hamiltonian system (1.1) has a Smale horseshoe in its dynamics on the energy surface \( H = H_\alpha \).

Now we assume that \( M_1(t_0) \equiv 0 \). As stated in Sect. 1, this assumption holds for (1.3) with \( c/d = 3/4 \) (see also Sect. 5). Let

\[ K(v) = \int_0^\infty [q_1(x^h(t), \Psi(t)v) - q_1(0, \Phi(t)v)] dt \]

\[ + \int_{-\infty}^0 [q_1(x^h(t), \Psi(t)v) - q_1(0, \Phi(t)v)] dt. \]

Define the second-order Melnikov function as

\[ M_2(t_0) = -\Phi(t_0)e_1 \cdot D_x^2 H(0,0) B_- JD_v K(B_-^{-1}\Phi(t_0)e_1) \]

\[ + q_1(0, \Phi(t_0)e_1) - q_1(0, B_0 \Phi(t_0)e_1), \]

where “•” represents the inner product. We state our main result as follows (see Sect. 4 for the proof).

**Theorem 3.2.** Suppose that \( M_1(t_0) \equiv 0 \) and that \( M_2(t_0) \) has a simple zero. Then the conclusion of Theorem 3.1 still holds.

**Remark 3.3.** For \( q_1(x, \eta) \equiv 0 \), we can obtain a result similar to Theorem 3.2 by computing higher-order terms of \( O(|y|^4) \) although more tedious computations are needed.
The proof is very similar to that of Lemma 3.2 of [33]. We first note that for \( \varepsilon, \delta \in \mathbb{R}^2 \) to lemma [12], we easily obtain the first part. From the smoothness of \( x, y, z \) by Proposition 2.3 there exists a periodic orbit \( (x, y) = (x^h(t), 0) \),

\[
x = x^h(0) + O(\varepsilon^2), \quad y = \varepsilon y_0,
\]

at \( t = 0 \) for any \( \varepsilon_0 \neq 0 \) \( \in \mathbb{R}^2 \), such that the following expressions hold with uniform validity in the indicated time intervals:

\[
x_\varepsilon^x(t; \varepsilon_0) = x^h(t) + \varepsilon^2 x_\varepsilon^x(t; \varepsilon_0), \quad y_\varepsilon^x(t; \varepsilon_0) = \varepsilon y_\varepsilon^x(t; \varepsilon_0)
\]

for \( t \in [0, t_1] \) with any \( t_1 > 0 \);

\[
x_\varepsilon^y(t; \varepsilon_0) = x^h(t) + \varepsilon^2 x_\varepsilon^y(t; \varepsilon_0), \quad y_\varepsilon^y(t; \varepsilon_0) = \varepsilon y_\varepsilon^y(t; \varepsilon_0)
\]

for \( t \in [-t_2, 0] \) with any \( t_2 > 0 \). See Fig. 2. Here \( (x_\varepsilon^x(t; \varepsilon_0), y_\varepsilon^x(t; \varepsilon_0)) \) are solutions to

\[
\dot{x} = JD^2_x H(x^h(t), 0) + \varepsilon p_1(x^h(t), \eta) + \varepsilon^2 p_2(x^h(t), \eta) + O(\varepsilon^2),
\]

\[
\dot{y} = JD^2_y H(x^h(t), 0) + \varepsilon q_1(x^h(t), \eta) + \varepsilon^2 q_2(x^h(t), \eta) + O(\varepsilon^2),
\]

and satisfy

\[
JD_x H(x^h(0), 0) \xi + \varepsilon q_1(0, \eta) = 0
\]

and \( y_\varepsilon^y(0; \varepsilon_0) = \varepsilon y_0 \). In addition, \( z_\varepsilon^{\alpha}(t; \varepsilon_0) \) have energy

\[
H(z_\varepsilon^{\alpha}(t; \varepsilon_0)) = \varepsilon^2 q_0(0, B \pm \eta_0) + O(\varepsilon^3).
\]

**Proof.** The proof is very similar to that of Lemma 3.2 of [33]. We first note that by Proposition 2.3 there exists a periodic orbit \( \gamma^\alpha \in \mathcal{M} \) for \( \varepsilon > 0 \) sufficiently small, such that orbits on \( W^{s,u}(\mathcal{M}) \) passing through the point given by (4.1) are contained in \( W^{s,u}(\gamma^\alpha) \). Substituting (4.2) and (4.3) into (1.1) and using Gronwall’s lemma [12], we easily obtain the first part. From the smoothness of \( x_\varepsilon^{\alpha}(t; \varepsilon_0) \) on \( \varepsilon \) we can choose \( \xi_\varepsilon^{\alpha}(0; \varepsilon_0) \) such that they satisfy (4.5).
Let
\[ \bar{H}(\xi, \eta, t) = D_x H(x^b(t), 0)\xi + \frac{1}{2} D^2_\eta H(x^b(t), 0)(\eta, \eta). \]
Then we have
\[ H(z^a_c(t; \eta_0)) = \epsilon^2 \bar{H}(z^a_c(t; \eta_0), \eta^a_c(t; \eta_0), t) + \mathcal{O}(\epsilon^2) \]
since \( H(x^b(t), 0) = 0 \) and \( D_\eta H(x, 0) = 0 \). Using (4.5), we obtain
\[ H(\xi^a_c(t; \eta_0), \eta^a_c(t; \eta_0), t) = q_0(0, B_\pm \eta_0) + \mathcal{O}(\epsilon). \]
which yields (4.6).

Let \( z^a_c(t; \eta_0) = (x^a_c(t; \eta_0), y^a_c(t; \eta_0)) \) be orbits on \( W^{s.a}(\mathcal{M}) \) in Lemma 4.1 such that \( y^a_c(0; \eta_0) = y^a_c(0; \eta_0) = c_{\eta_0} \). Note that one may have \( \lim_{t \to \infty} z^a_c(t; \eta_0) \notin \mathcal{M} \) and \( H(z^a_c(t; \eta_0)) \neq H(z^a_c(t; \eta_0)) \). Denote by \( d(\eta_0, \epsilon) \) the distance between \( z^a_c(t; \eta_0) \) and \( z^a_c(t; \eta_0) \) along the direction \( (D_x H(x^b(0), 0), 0) \) near the point \( (x, y) = (x^b(0), \eta_0) \). We have
\[ d(\eta_0, \epsilon) = \frac{D_x H(x^b(0), 0)}{|D_x H(x^b(0), 0)|} \cdot (x^a_c(0; \eta_0) - x^a_c(0; \eta_0)), \]
which can also measure the distance between \( W^{s}(\mathcal{M}) \) and \( W^{u}(\mathcal{M}) \) (see Fig. 2).

Let
\[ \Delta^a_c(t; \eta_0) = D_x H(x^b(t), 0) \cdot \xi^a_c(t; \eta_0). \]
From Lemma 4.1 we see that
\[ d(\eta_0, \epsilon) = \epsilon^2 c_0(\Delta^a_c(0; \eta_0) - \Delta^a_c(0; \eta_0)), \quad (4.7) \]
where
\[ c_0 = |D_x H(x^b(0), 0)|^{-1} \neq 0. \]

Employing “Melnikov’s trick” as in the standard Melnikov technique (e.g., Sect. 4.3 of [12]) and noting that \( D_x H(x^b(t), 0) \) exponentially tends to zero as \( t \to \pm \infty \) and \( |\xi^a_c(t; \eta_0)| < \infty \) (resp. \( |\xi^a_c(t; \eta_0)| = \infty \) for \( t \in [0, \infty) \) (resp. \( t \in (-\infty, 0) \)), we obtain
\[
\begin{align*}
\Delta^a_c(0; \eta_0) &= -\int_0^\infty D_x H(x^b(t), 0) \cdot [p_1(x^b(t), \eta^a_c(t; \eta_0))] dt + \mathcal{O}(\epsilon^2), \\
\Delta^a_c(0; \eta_0) &= \int_{-\infty}^0 D_x H(x^b(t), 0) \cdot [p_1(x^b(t), \eta^a_c(t; \eta_0))] dt + \mathcal{O}(\epsilon^2).
\end{align*}
\]
Here we used (4.4) and the fact that \( t_1 \) and \( t_2 \) can be chosen by arbitrarily large values.

On the other hand, we can take \( t_0 \in \mathbb{R} \) such that \( \Phi(t_0)e_1 = B_+ \eta_0/|\eta_0| \) since \( |B_- \eta_0| = |\eta_0| \). Hence, we estimate the solutions \( \eta^a_c(t; \eta_0) \) of (4.4) to obtain
\[ \eta^a_c(t; \eta_0) = |\eta_0| \Psi(t) B^{-1}_+ \Phi(t_0)e_1 + \mathcal{O}(\epsilon). \]
Since by Lemma 3.4 of [33] one can rewrite
\[ M_1(t_0) = \int_{-\infty}^\infty D_x H(x^b(t), 0) \cdot p_1(x^b(t), \Psi(t) B^{-1}_+ \Phi(t_0)e_1) dt, \]
it follows from (4.8) that
\[ \Delta^{\upsilon}(0; \eta_0) - \Delta^{\epsilon}(0; \eta_0) \]
\[ = \int_{-\infty}^{\infty} D_x H(x^h(t), 0) \cdot p_1(x^h(t), |\eta_0| \Psi(t)B_{-1}^{-1}\Phi(t_0)e_1) dt + O(\epsilon) \]
\[ = |\eta_0|^2 M_1(t_0) + O(\epsilon) \quad (4.9) \]
since \( q_0(0, \eta) \) is a quadratic form of \( \eta \) (see also Lemma 3.3 of [33]).

Now we assume that \( M_1(t_0) \equiv 0 \). For any nonzero vector \( \eta \in \mathbb{R}^2 \)
\[ B_+^T D_0^2 H(0, 0) B_+ = B_+^T D_0^2 H(0, 0) B_- \quad (4.10) \]
since choosing \( t_0 \) such that \( \Phi(t_0)e_1 = B_- \eta/|\eta| \), we have
\[ M_1(t_0) = \frac{1}{2}[\Phi(t_0)e_1 \cdot D_0^2 H(0, 0)\Phi(t_0)e_1 - B_0\Phi(t_0)e_1 \cdot D_0^2 H(0, 0)B_0\Phi(t_0)e_1] \]
\[ = \frac{1}{2|\eta|^2}[B_-\eta \cdot D_0^2 H(0, 0) B_- \eta - B_+\eta \cdot D_0^2 H(0, 0) B_+ \eta] = 0. \]

Denote by \( \bar{B}_0 \) the matrix given by (4.10). Letting \( \eta = \Psi(t)u \) in the second equation of (4.4), we have
\[ \dot{u} = \epsilon \Psi^{-1}(t) JD_{\eta} q_1(x^h(t), \Psi(t)u) \]
up to \( O(\epsilon^2) \). As in the averaging method [12, 27], using the near-identity transformation
\[ u = v + \epsilon w^\pm(v, t), \]
we obtain
\[ \dot{v} = \epsilon JD_v q_1^\pm(v) \]
up to \( O(\epsilon^2) \), where
\[ JD_v q_1^\pm(v) = \lim_{T \to \pm\infty} \frac{1}{T} \int_{-T}^{T} \Psi^{-1}(t) JD_{\eta} q_1(x^h(t), \Psi(t)v) dt \]
\[ = \lim_{T \to \pm\infty} \frac{1}{T} \int_{-T}^{T} B_{\pm}^{-1}\Phi(-t) JD_{\eta} q_1(0, \Phi(t)B_{\pm}v) dt, \quad (4.11) \]
\[ w^\pm(v, t) = \int_{0}^{t} [\Psi^{-1}(t) JD_{\eta} q_1(x^h(t), \Psi(t)v) - JD_{v} q_1^\pm(v)] dt. \]

Since \( D_{\eta} q_1(0, \eta) \) is a quadratic form of \( \eta \) and consequently the integrand in the first equation of (4.11) consists of cubic functions of \( \sin \omega t \) and \( \cos \omega t \), we have
\[ JD_v q_1^\pm(v) \equiv 0, \quad \text{so that} \]
\[ \dot{v} = O(\epsilon^2) \quad (4.12) \]
and
\[ w^\pm(v, t) = w(v, t) = \int_{0}^{t} \Psi^{-1}(t) JD_{\eta} q_1(x^h(t), \Psi(t)v) dt. \quad (4.13) \]

Hence, Eq. (4.9) is expressed as
\[ \Delta^{\upsilon}(0; \eta_0) - \Delta^{\epsilon}(0; \eta_0) \]
\[ = \epsilon \left\{ \int_{-\infty}^{\infty} D_x H(x^h(t), 0) \cdot [D_{\eta} p_1(x^h(t), \Psi(t)\eta_0)\Psi(t)w(\eta_0, t) \right. \]
\[ + p_2(x^h(t), \Psi(t)\eta_0)] dt \} + O(\epsilon^2) \quad (4.14) \]
since the solution (4.12) with \( v(0) = \eta_0 \) is given by

\[
v = \eta_0 + O(e^2)
\]

for \( t = O(1) \).

**Lemma 4.2.** We have

\[
\frac{d}{dt} \left[ D_\eta q_0(x^b(t), \Psi(t)v) \cdot \Psi(t)w(v, t) + q_1(x^b(t), \Psi(t)v) \right] = -D_x H(x^b(t), 0) \cdot \left[ D_\eta q_1(x^b(t), \Psi(t)v) \Psi(t)w(v, t) + p_2(x^b(t), \Psi(t)v) \right].
\]

(4.15)

**Proof.** We compute

\[
\frac{d}{dt} \left[ D_\eta q_0(x^b(t), \Psi(t)v) \cdot \Psi(t)w(v, t) + q_1(x^b(t), \Psi(t)v) \right] = \left[ D_x D_\eta q_0(x^b(t), \Psi(t)v) \right] \dot{x^b(t)} \\
+ D_\eta q_0(x^b(t), \Psi(t)v) \cdot \left[ \dot{\Psi}(t)w(v, t) + \Psi(t) \frac{\partial w}{\partial t}(v, t) \right] \\
+ D_x q_1(x^b(t), \Psi(t)v) \cdot \dot{x^b(t)} + D_\eta q_1(x^b(t), \Psi(t)v) \cdot \dot{\Psi}(t)v.
\]

Substituting

\[
\dot{x^b(t)} = JD_x H(x^b(t), 0), \\
\dot{\Psi}(t) = JD^2_y H(x^b(t), 0) \Psi(t), \\
D_\eta q_0(x^b(t), \eta) = D^2_y H(x^b(t), 0) \eta, \\
\frac{\partial w}{\partial t}(v, t) = \Psi^{-1}(t)JD_\eta q_1(x^b(t), \Psi(t)v)
\]

into the above equation and using the relations \( p_j(x, \eta) = JD_x q_j(x, \eta), j = 1, 2, \) and \( J\eta \cdot \eta' = -\eta \cdot J\eta' \) for \( \eta, \eta' \in \mathbb{R}^2 \), we obtain (4.15).

\( \square \)

**Lemma 4.3.** We have

\[
D^2_y H(0, 0) B_\pm v \cdot J \int_0^t \Phi^T(t) D_\eta q_1(0, \Phi(t)B_\pm v) dt = -q_1(0, \Phi(t)B_\pm v) + q_1(0, B_\pm v).
\]

(4.16)

**Proof.** We compute

\[
\frac{d}{dt} [q_1(0, \Phi(t)B_\pm v)] = \dot{\Phi}(t)B_\pm v \cdot D_\eta q_1(0, \Phi(t)B_\pm v) \\
= JD^2_y H(0, 0) \Phi(t)B_\pm v \cdot D_\eta q_1(0, \Phi(t)B_\pm v).
\]

Since \( \Phi(t) \) is a solution of (3.1), we have

\[
JD^2_y H(0, 0) \Phi(t) = \Phi(t)JD^2_y H(0, 0),
\]

so that by \( J^T = -J \)

\[
[JD^2_y H(0, 0) \Phi(t)]^T = -D^2_y H(0, 0) J \Phi^T(t).
\]

Hence,

\[
\frac{d}{dt} [q_1(0, \Phi(t)B_\pm v)] = -D^2_y H(0, 0) B_\pm v \cdot J \Phi^T(t) D_\eta q_1(0, \Phi(t)B_\pm v)
\]

from which Eq. (4.16) immediately follows by \( \Phi(0) = \text{id} \).

\( \square \)
Integrating (4.15) from \( t = -\infty \) to \( \infty \) and using (3.3), (4.14) and the relation
\[
(\Phi(t)B_-)^T \mathcal{D}_{\eta}^2 H(0,0) \Phi(t)B_- = (\Phi(t)B_+)^T \mathcal{D}_{\eta}^2 H(0,0) \Phi(t)B_+ = \dot{B}_0,
\]
we obtain
\[
\Delta^u_\eta(0; \eta_0) - \Delta^s_\eta(0; \eta_0)
= \epsilon \left\{ \lim_{t \to -\infty} [\eta_0 \cdot \dot{B}_0 w(\eta_0, t) + q_1(0, \Phi(t)B_- \eta_0)]
- \lim_{t \to \infty} [\eta_0 \cdot \dot{B}_0 w(\eta_0, t) + q_1(0, \Phi(t)B_+ \eta_0)] \right\} + O(\epsilon^2).
\]
(4.17)

Since \( B_\pm \) are symplectic, we have \( B_\pm J B_\pm^T = J \) (see, e.g., Theorem II.A.2 of [21]) so that
\[
\dot{B}_0 J B_\pm^T = B_\pm J \mathcal{D}_{\eta}^2 H(0,0) B_\pm J B_\pm^T = B_\pm J \mathcal{D}_{\eta}^2 H(0,0).J.
\]
Hence, by Lemma 4.3,
\[
\eta_0 \cdot \dot{B}_0 J B_\pm^T \int_0^t \Phi^T(t) \mathcal{D}_{\eta} q_1(0, \Phi(t)B_\pm \eta_0) dt
= -q_1(0, \Phi(t)B_\pm \eta_0) + q_1(0, B_\pm \eta_0).
\]
Noting that \( \Psi(t) \) is symplectic and using (4.13) and the relation \( \Psi^{-1}(t) = -J \Psi(t) J \) (see, e.g., Theorem II.A.2 of [21]), we obtain
\[
\eta_0 \cdot \dot{B}_0 w(\eta_0, t) + q_1(0, \Phi(t)B_\pm \eta_0)
= \eta_0 \cdot \dot{B}_0 J \int_0^t [\Psi^T(t) \mathcal{D}_{\eta} q_1(x(t), \Psi(t)\eta_0)] dt + q_1(0, B_\pm \eta_0).
\]
(4.18)

Using (4.18) in (4.17), we have
\[
\Delta^u_\eta(0; \eta_0) - \Delta^s_\eta(0; \eta_0)
= \epsilon [-B_- \eta_0 \cdot \mathcal{D}_{\eta}^2 H(0,0) B_- J \mathcal{D}_{\eta} K(\eta_0) + q_1(0, B_- \eta_0) - q_1(0, B_+ \eta_0)] + O(\epsilon^2)
\]
(4.19)
since
\[
\mathcal{D}_{\eta} q_1(0, \Psi(t) v) = \Psi^T(t) \mathcal{D}_{\eta} q_1(0, \Psi(t) v).
\]

We are now in a position to give a proof of Theorem 3.2.

**Proof of Theorem 3.2.** Suppose that the hypotheses of Theorem 3.2 hold. Assume that \( |\eta_0| = 1 \) without loss of generality and let \( \eta_0 = B^{-1} \Phi(t_0) e_1 \). Then equation (4.19) becomes
\[
\Delta^u_\eta(0; \eta_0) - \Delta^s_\eta(0; \eta_0) = \epsilon M_2(t_0) + O(\epsilon^2),
\]
so that by (4.7)
\[
d(t_0, \epsilon) = \epsilon^3 c_0 M_2(t_0) + O(\epsilon^4).
\]
Application of the implicit function theorem to
\[
\frac{1}{\epsilon^3} d(t_0, \epsilon) = 0
\]
shows that \( W^s(\mathcal{M}) \) and \( W^u(\mathcal{M}) \) intersect near
\[
(x, y) = (x(t_0), \epsilon B^{-1} \Phi(t_0) e_1).
\]
Moreover, using (4.7), we obtain
\[
\frac{\partial}{\partial t_0} d(t_0, \epsilon) = c_0 D_N H(x^b(0), 0) \cdot \left( \frac{\partial x^u}{\partial t_0}(0; t_0) - \frac{\partial x^s}{\partial t_0}(0; t_0) \right)
\]
\[= \epsilon^3 c_0 \frac{dM_2}{dt_0}(t_0) + O(\epsilon^4), \]
which means that the intersection between $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ is transversal on the energy surface $H = \epsilon^2 q_0(\Phi(t_0)e_1) + O(\epsilon^3)$. Thus, we complete the proof of Theorem 3.2. \hfill \Box

5. **An example.** In this section we apply our theory to the Hamiltonian (1.3) with $\beta_2/\beta_1 = 1/3$, which corresponds to the Hénon-Heiles Hamiltonian (1.2) with $c/d = 3/4$. Eq. (2.1) has a homoclinic orbit given by (2.2). The NVEs (3.1) and (3.2) become
\[
\dot{\eta}_1 = \eta_2, \quad \dot{\eta}_2 = -\omega \eta_1 \tag{5.1}
\]
and
\[
\dot{\eta}_1 = \eta_2, \quad \dot{\eta}_2 = -\left( \omega^2 + \frac{1}{2} \text{sech}^2 \left( \frac{t}{2} \right) \right) \eta_1, \tag{5.2}
\]
respectively. The fundamental matrices to (5.1) and (5.2) are given by
\[
\Phi(t) = \begin{pmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{pmatrix} \quad \text{and} \quad \Psi(t) = \begin{pmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{pmatrix},
\]
respectively, where
\[
\Psi_{11}(t) = \frac{1}{2\omega} \left( 2\omega \cos \omega t - \sin \omega t \tanh \frac{t}{2} \right),
\]
\[
\Psi_{12}(t) = \frac{2}{4\omega^2 + 1} \left( 2\omega \sin \omega t + \cos \omega t \tanh \frac{t}{2} \right),
\]
\[
\Psi_{21}(t) = -\frac{1}{4\omega} \left( 4\omega^2 \sin \omega t + 2\omega \cos \omega t \tanh \frac{1}{2} t + \sin \omega t \text{sech}^2 \frac{1}{2} t \right),
\]
\[
\Psi_{22}(t) = -\frac{1}{4\omega^2 + 1} \left( 4\omega^2 \cos \omega t - 2\omega \sin \omega t \tanh \frac{1}{2} t + \cos \omega t \text{sech}^2 \frac{1}{2} t \right).
\]
We remark that the fundamental matrix to (5.2) can be expressed in terms of elementary functions since $M_1(t_0) \equiv 0$, which implies, as shown in [36], that the normal variational equation (5.2) is integrable in the meaning of the differential Galois theory [23, 25]. We compute
\[
B_{\pm} = \begin{pmatrix} 1 & \pm \frac{2}{4\omega^2 + 1} \\ \frac{1}{2} & \frac{4\omega^2}{4\omega^2 + 1} \end{pmatrix}, \quad B_0 = \begin{pmatrix} 4\omega^2 - 1 & 4 \\ 4\omega^2 + 1 & 4\omega^2 \end{pmatrix}.
\]
On the other hand,
\[
q_0(x, \eta) = \frac{1}{2} [(\omega^2 + \beta_2 x_1) \eta_1^2 + \eta_2^2], \quad q_1(x, \eta) = \frac{1}{3} \beta_3 \eta_1^3.
\]
We easily see that
\[
M_1(t_0) = q_0(0, \Phi(t_0)e_1) - q_0(0, B_0 \Phi(t_0)e_1) \equiv 0.
\]
After a lengthy calculation, we obtain

$$K(v) = \frac{1}{3} \beta_3 [K_{30}(\omega)v_1^3 + K_{12}(\omega)v_1v_2^2],$$

where

$$K_{30}(\omega) = \frac{6\omega^2 + 1}{24\omega^4} (4 - 3\pi\omega(3 \cosh 2\pi\omega + 1) \csch 3\pi\omega),$$

$$K_{12}(\omega) = \frac{4 - 6\pi\omega(6\omega^2 + 1)(\cosh 2\pi\omega + 1) \csch 3\pi\omega}{\omega^2(4\omega^2 + 1)^2}.$$ 

So we compute (3.7) as

$$M_2(t_0) = \beta_3 (\cos \omega t_0 + 2\omega \sin \omega t_0)$$

$$\times \{ K_0(\omega) + K(\omega) [(4\omega^2 - 1) \cos 2\omega t_0 - 4\omega \sin 2\omega t_0] \},$$

where

$$K_0(\omega) = \frac{4(12\omega^2 - 1) - 48\omega^4 K_{30}(\omega) + \omega^2(4\omega^2 + 1)^2 K_{12}(\omega)}{12(4\omega^2 + 1)^2},$$

$$K_1(\omega) = \frac{4(12\omega^2 - 1) - 48\omega^4 K_{30}(\omega) + 3\omega^2(4\omega^2 + 1)^2 K_{12}(\omega)}{12(4\omega^2 + 1)^3}.$$ 

If $K_0(\omega) = K_1(\omega) = 0$, then

$$K_{12}(\omega) = \frac{4}{\omega^2(4\omega^2 + 1)^2},$$

i.e.,

$$6\pi\omega(6\omega^2 + 1)(\cosh(2\pi\omega) + 1) \csch(3\pi\omega) = 0,$$

which never holds for $\omega > 0$. Hence, for any $\omega > 0$, $K_0(\omega) \neq 0$ or $K_1(\omega) \neq 0$. Thus, the second-order Melnikov function $M_2(t_0)$ has zeros at

$$\omega_0 t_0 = -\arctan \left( \frac{1}{2\omega} \right) \quad \text{and} \quad \pi - \arctan \left( \frac{1}{2\omega} \right). \quad (5.3)$$

We easily see that

$$(4\omega^2 - 1) \cos 2\omega t_0 - 4\omega \sin 2\omega t_0 = 4\omega^2 + 1$$

at the zeros of (5.3) and

$$K_0(\omega) + (4\omega^2 + 1) K_1(\omega) = \frac{24\pi\omega(6\omega^2 + 1) \sinh(\pi\omega)}{2 \cosh(2\pi\omega) + 1} \neq 0$$

for $\omega > 0$. Thus, the zeros of (5.3) for $M_2(t_0)$ are simple. So we prove the following result.

**Theorem 5.1.** Let $\beta_2/\beta_1 = 1/3$ and $\beta_3 \neq 0$. Then for $\alpha > 0$ sufficiently small the Hamiltonian system (1.1) with the Hamiltonian (1.3) has orbits transversely homoclinic to the periodic orbit $\gamma^\alpha$ near the saddle-center $O$ and a Smale horseshoe in its dynamics on the energy surface $H = H_\alpha$. In particular, this statement holds for the Hénon-Heiles Hamiltonian (1.2) with $c/d = 3/4$.

Figure 3 shows projections of numerically computed periodic orbits near the saddle-center $O$ on the $y$-plane for the Hamiltonian (1.3), where the parameter values were given by (1.4) with $c = 0.75$ and $d = 1$. To obtain the periodic orbits in
Fig. 3, we considered a boundary value problem for (1.1) with (1.3) under boundary conditions

\[ x_1(0) = x_1(T), \quad x_2(0) = x_2(T), \quad y_1(0) = y_1(T) = 0, \]

where \( T \) is a period, and numerically continued a solution starting at the origin for \( H = 0 \) using AUTO [5]. Figure 4 shows numerically computed stable and unstable manifolds of the periodic orbit near the saddle-center with \( H = 3 \times 10^{-3} \) on the Poincaré section \( \Sigma = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 | y_1 = 0, y_2 > 0\} \). Here a computer software called Dynamics [26] was used with the assistance of a differential equation solver called DOP853 [13], which is based on the explicit Runge-Kutta method of order 8 [6] and has a fifth order error estimator with third order correction and a dense output of order 7. To obtain a point at which a computed trajectory intersects the Poincaré section, an interval \([t_{k-1}, t_k] \) of numerical integration such that \( y_1(t_{k-1}) < 0 \) and \( y_1(t_k) \geq 0 \) was searched and the method of bisection was used for the interval with an error of \( 10^{-8} \). A tolerance of \( 10^{-8} \) was chosen in our computation. See also [38, 39] for more details on the incorporation of DOP853 in Dynamics, which is included in a package of AUTO and Dynamics drivers called HomMap. From Fig. 4 we see that the stable and unstable manifolds of the periodic orbit intersect transversely as predicted by Theorem 5.1.

REFERENCES

Figure 4. Numerically computed stable and unstable manifolds of a periodic orbit near the saddle-center $O$ for the Hamiltonian (1.3), where the parameter values were chosen via (1.4) with $c = 0.75$ and $d = 1$, and the Hamiltonian energy is $H = 3 \times 10^{-3}$.


