On Markov processes

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2016 January 4-8

Markov processes

Let T be a hitting time defined by

$$T_x := \min\{n \geq 1; X_n = x\}$$

where $\min \emptyset = \infty$.
We are interested in the mean return time $E[T_x|X_0 = x]$. If return probability $P(T_x < \infty|X_0 = x)$ is strictly less than 1, then mean return time is ∞.

Definition (recurrent, transient)

We call the state $x \in S$ is recurrent iff

$$P(T_x < \infty|X_0 = x) = 1.$$

We also call the state x is transient iff the state x is not recurrent.

Markov processes

Let $S = \{a, b, c\}$ be a state space. Suppose that transition matrix is given by

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/3 & 2/3 \end{pmatrix}$$

Then we have

$$\begin{aligned}
&P(T_a = 1|X_0 = a) = P(X_1 = a|X_0 = a) = \frac{1}{2} \\
&P(T_a = 2|X_0 = a) = P(X_2 = a, X_1 \neq a|X_0 = a) \\
&= P(X_2 = a|X_1 = b)P(X_1 = b|X_0 = a) \\
&\quad + P(X_2 = a|X_1 = c)P(X_1 = c|X_0 = a) = 0
\end{aligned}$$

Similarly, we have

$$P(2 \leq T_a < \infty|X_0 = a) = 0.$$

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Similarly, we have

$$
\begin{aligned}
&P(T_b = 1|X_0 = b) = P(X_1 = b|X_0 = b) = \frac{1}{2} \\
&P(T_b = 2|X_0 = b) = P(X_2 = b, X_1 \neq b|X_0 = b) \\
&= P(X_2 = b|X_1 = a)P(X_1 = a|X_0 = b) \\
&\quad + P(X_2 = b|X_1 = c)P(X_1 = c|X_0 = b) = \frac{1}{6}
\end{aligned}
$$

Inductively, for $k \geq 2$ we have

$$
P(T_b = k|X_0 = b) = \frac{1}{6}\left(\frac{2}{3}\right)^{k-2}
$$

Markov processes

Hence we have

$$P(T_b < \infty | X_0 = b) = \sum_{k=1}^{\infty} (T_b = k | X_0 = b)$$

$$= \frac{1}{2} + \frac{1}{6} \sum_{k=2}^{\infty} \left(\frac{2}{3} \right)^{k-2} = 1$$

In fact, we only take care of bottom right 2×2 submatrix. If a transition matrix is large and complicated, then it seems hard to compute a return probability directly.

Markov processes

Theorem (recurrent)

The state x is recurrent iff

$$\sum_{n=0}^{\infty} P(X_n = x | X_0 = x) = \infty$$

Proof

We set

$$P_n = P(X_n = x | X_0 = x), \quad f_n = P(T_x = n | X_0 = x)$$

Markov processes

We divide an event $\{X_n = x\}$ by means of first return time $T_x = k$. We have

$$\{X_n = x\} = \bigcup_{k=1}^{n} \{X_n = x, T_x = k\}$$

Note that these events in the right hand side is mutually exclusive. We can rewrite as

$$\{T_x = k\} = \{X_k = x, X_{k-1} \neq x, X_{k-2} \neq x, \ldots, X_1 \neq x\}$$

Markov processes

By using Markov property, we have

$$
\begin{aligned}
P_n &= P(X_n = x | X_0 = x) \\
&= P(\bigcup_{k=1}^{n} P(\{X_n = x, T_x = k\} | X_0 = x) \\
&= \sum_{k=1}^{n} P(X_n = x, X_k = x, X_{k-1} \neq x, X_{k-2} \neq x, \dots, X_1 \neq x | X_0 = x) \\
&= \sum_{k=1}^{n} P(X_n = x | X_k = x) \\
&\qquad \times P(X_k = x, X_{k-1} \neq x, X_{k-2} \neq x, \dots, X_1 \neq x | X_0 = x) \\
&= \sum_{k=1}^{n} P(X_n = x | X_k = x) P(T_x = k | X_0 = x) = \sum_{k=1}^{n} P_{n-k} f_k
\end{aligned}
$$

Namely $P_n = \sum_{k=1}^{n} P_{n-k} f_k$ for $n \geq 1$. Since $P_0 = 1$, we sum over n, exchange the order of summation and taking change of variables, we have

$$\begin{aligned}\sum_{n=0}^{\infty} P_n &= 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} P_{n-k} f_k \\ &= 1 + \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} P_{n-k} f_k = 1 + \sum_{k=1}^{\infty} f_k \sum_{n=0}^{\infty} P_n\end{aligned}$$

Hence we formally have

$$\sum_{k=1}^{\infty} f_k = 1 - \frac{1}{\sum_{n=0}^{\infty} P_n}$$

This expression coincides with the statement of the theorem.

Markov processes

In fact, the statement "if $\sum P_n < \infty$ then the state x is transient" is correct. But if $\sum P_n = \infty$, we can only claim is $\sum f_n \neq 0$.
In this case, we use the generating function.
Suppose that $|s| < 1$. Then we have similar expression as

$$\sum_{n=0}^{\infty} P_n s^n = 1 + \sum_{n=1}^{\infty}(\sum_{k=1}^{n} P_{n-k} f_k) s^n = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} P_{n-k} s^{n-k} f_k s^k$$

$$= 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} P_{n-k} s^{n-k} f_k s^k = 1 + \sum_{k=1}^{\infty} f_k s^k \sum_{n=0}^{\infty} P_n s^n$$

Since $|s| < 1$, we have

$$\sum_{k=1}^{\infty} f_k s^k = 1 - \frac{1}{\sum_{n=0}^{\infty} P_n s^n}$$

We take a limit as $s \to 1$. Then we have verified the case $\sum P_n = \infty$. □

Problem

Why we can take such a limit? (or why we can exchange the limit and infinite sum?) Give some explanation.

Markov processes

Theorem (relation between recurrent and equivalence class)

If the states $x \leftrightarrow y$ and the state x is recurrent, then the state y is also recurrent.

Proof

Since $x \leftrightarrow y$, we have

$$\exists n_1, n_2 \text{ s.t. } P(X_{n_1} = y | X_0 = x) > 0, \; P(X_{n_2} = x | X_0 = y) > 0.$$

By using Markov property, we have

$$\begin{aligned}
&P(X_{n_2+n+n_1} = y | X_0 = y) \\
&= \sum_{z,w} P(X_{n_2+n+n_1} = y, X_{n+n_1} = z, X_{n_1} = w | X_0 = y) \\
&\geq P(X_{n_2+n+n_1} = y, X_{n+n_1} = x, X_{n_1} = x | X_0 = y) \\
&= P(X_{n_2} = y | X_0 = x) P(X_n = x | X_0 = x) P(X_{n_1} = x | X_0 = y)
\end{aligned}$$

Markov processes

By applying the Theorem (recurrent), we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} P(X_k = y|X_0 = y) \\
&\geq \sum_{n=0}^{\infty} P(X_{n_2+n_1+n} = y|X_0 = y) \\
&\geq \sum_{n=0}^{\infty} P(X_{n_2} = y|X_0 = x)P(X_n = x|X_0 = x)P(X_{n_1} = x|X_0 = y) \\
&= \infty
\end{aligned}$$

□

Markov processes

Corollary (transient and equivalent class)

Suppose that $x \leftrightarrow y$ and x is transient, then y is also transient.

Problem

Prove this Corollary.

Proposition (recurrent and equivalent class)

Suppose that x is recurrent and $x \to y$, then we have $y \to x$.

<u>Proof</u>

Since we suppose that $x \to y$, we have $\{n; P(X_n = y | X_0 = x) > 0\} \neq \emptyset$. Hence we set

$$n = \min\{n; P(X_n = y | X_0 = x) > 0\},$$

namely, we have $\{x_1, x_2, \dots, x_{n-1}\} \cap \{x, y\} = \emptyset$ and

$$P(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 | X_0 = x) > 0.$$

By proving following claim, we prove our proposition;
claim: Suppose that $y \not\to x$ then x is transient.
Suppose that $y \not\to x$, then we have $\forall m \geq 1$,

$$P(X_m \neq x, X_{m-1} \neq x, \dots X_1 \neq 1 | X_0 = y) = 1.$$

Hence we have $\forall m \geq 1$,

$$\begin{aligned} &P(X_{n+m} \neq x, X_{n+m-1} \neq x, \dots, X_{n+1} \neq x, \\ &\quad X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 | X_0 = x) \\ &= P(X_{n+m} \neq x, X_{n+m-1} \neq x, \dots, X_{n+1} \neq x | X_n = y) \\ &\quad P(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 | X_0 = x) \\ &= P(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 | X_0 = x) > 0 \end{aligned}$$

(Namely we can estimate this probability below by positive constant which does not depend on m.)

We define $F_N = \bigcap_{n=1}^{N}\{X_n \neq x\}$. Then F_N is decreasing sequence and

$$\{T_x = \infty\} = (\bigcup_{n\geq 1}\{X_n = x\})^c = \bigcap_{n\geq 1}\{X_n = x\}^c = \lim F_N$$

Hence we have

$$\begin{aligned}
&P(T_x = \infty, X_n = y | X_0 = x) \\
&= P(\lim_{m\to\infty}\{X_{n+m} \neq x, \dots, X_{n+1} \neq x, X_n = y\} | X_0 = x) \\
&= \lim_{m\to\infty} P(X_{n+m} \neq x, \dots, X_{n+1} \neq x, X_n = y | X_0 = x) \\
&\geq P(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 | X_0 = x) > 0
\end{aligned}$$

Namely, x is transient. □

Markov processes

Corollary (recurrent and equivalent class)

Suppose that $\exists y$ s.t. $x \to y$ and $y \not\to x$, then x transient.

Problem

Prove this Corollary.

Markov processes

Summary

• x is transient iff $\sum_n P(X_n = x | X_0 = x) = \infty$
• Suppose that $x \leftrightarrow y$, then both x, y are recurrent or are transient.
• Suppose that $x \to y$ and x is trnasient, then $y \to x$
• Suppose that $\exists y$ s.t. $x \to y$ and $y \not\to x$, then x is transient.