On Markov processes

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Markov processes

The Markov processes are one of the most tractable one in the stochastic processes. In order to apply some real phenomenon, Markov property is compatible.
The Markov processes have a lot of interesting properties. These properties allow to treat this process several ways. One of the interesting treatment is using the linear algebra or matrices theory.
The author is interesting in the application of the Markov processes. One of the most interested issue is the scaling limit of some family of Markov processes. In this issue, how to determine the space-time scaling ratio is one of the difficult question. One of the answer is that the order of the spectral gap gives such a ratio. If one knows the order of the spectral gap, then one can prove several results for example, hydrodynamic limit.
In this lecture we shall talk on the Markov processes and related matrices theories.

Markov processes

The conditional probability or conditional expectation is one of the most difficult objects in the probability theory. But in this lecture, we only consider following restricted form;

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) \neq 0.$$

It is sometimes convenient for us to define the conditional probability though if $P(B) = 0$. In this lecture it is written a conditional probability $P(A|B)$ for $P(B) = 0$, in this case, this conditional probability may not make sense. But in such case, there should be written like $P(A|B)P(B)$ and these are 0.
For notational convenience, we omit the symbol $\cap$, for example we write $P(A, B) = P(A \cap B)$.

Problem

If $P(B, C) > 0$, then show $P(A, B|C) = P(A|B, C)P(B|C)$.

Markov processes

Markov processes are subclass of the stochastic processes.
Stochastic processes are the random variables with time parameter.
Suppose that $(X_n)_{n\geq 0}$ be a stochastic process. Suppose that S be some set such that $X_n \in S$, then we say S a state space. (Usually, first we set a state space S and define a stochastic proceses X.)
A probability law of X_n for general stochastic process $(X_n)_n$ depends on the values of $X_0, X_1, \dots, X_{n-1}$.
If the stochastic process $(X_n)_{n\geq 0}$ satisfies following identity, then we call $(X_n)_{n\geq 0}$ (discrete time) Markov process;
If $P(X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \dots, X_1 = x_1, X_0 = x_0) > 0$, then

$$P(X_n = x_n | X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \dots, X_1 = x_1, X_0 = x_0)$$
$$= P(X_n = x_n | X_{n-1} = x_{n-1})$$

Namely the law of X_n is given by X_{n-1}.

Markov processes

As a terminology, there are Markov process and Markov chain. In this lecture we do not distinguish these terminology.
In the Mathematical dictionary, the difference between these are the cardinality of the state space. (Markov process has a state space with d-dimensional real space or Hausdoeff space and Markov chain has a state space at most countable set.)
In this lecture we only consider finite set for a state space. Some results are valid for some process with countable state space.

Markov processes

We can regard the Markov process as a natural extension of a random walk.
Random walker has his own coin (or dice) and for each time step, he toss the coin (throw the dice) and decide the direction and walk one step. Markov walker does not have his own coin (or dice), but for each point there is a coin (or dice). For each time, he use a coin (or dice) at his point and toss this coin (throw this dice) and decide the direction and walk one step. Hence we allow that the laws of coins are different each others.

Markov processes

If we retake a state space for large enough, then a lot of stochastic processes becomes Markov processes. If we only see the part of Markov process, then such a process does not satisfy the Markov property.

Example gain of double bet
We set a 2 dimensional stochastic process $(X_n, Y_n)_{n\geq 0}$ as follows; Let X_n be a gain of a bet with following strategy and Y_n be how much we bet for next time.

- It is convenient for us to set $X_0 = 0$.
- For our convenience, we are betting a coin tossing and if the result is H (head) (with probability p) then we win and if the result is T (tail) (with probability $1-p$) then we lose.
- We bet Y_n and if we win this game, then $X_{n+1} = X_n + Y_n$ and if we lose this game, then $X_{n+1} = X_n - Y_n$.
- As a strategy, if we win then we add bet for $Y_{n+1} = 2Y_n$ and if we lose then we reset for $Y_{n+1} = 1$.
- It is convenient for us to set $Y_0 = 1$.

In Japanese, such a strategy is called 倍賭け法

Then we can rewrite these setting as

$$(X_{n+1}, Y_{n+1}) = \begin{cases} (X_n + Y_n, 2Y_n) & \text{if H with probability } p \\ (X_n - Y_n, 1) & \text{if T with probability } 1-p \end{cases}$$

By this formulation, the process $(X_n, Y_n)_{n\geq 0}$ is a Markov process. Furthermore, if we only see the process $(Y_n)_{n\geq 0}$, then we have

$$Y_{n+1} = \begin{cases} 2Y_n & \text{with probability } p \\ 1 & \text{with probability } 1-p \end{cases}$$

hence it is also Markov process.

Markov processes

Let us consider two cases that the result of the coin tossing is 「HHT」「TTH」. In these cases, we can give the values of $(X_n, Y_n)_{n=0,1,2,3}$ as

$$
\begin{array}{lccccccc}
 & (X_0, Y_0) & & (X_1, Y_1) & & (X_2, Y_2) & & (X_3, Y_3) \\
\mathrm{HHT} & (0,1) & \to & (1,2) & \to & (3,4) & \to & (-1,1) \\
\mathrm{TTH} & (0,1) & \to & (-1,1) & \to & (-2,1) & \to & (-1,2)
\end{array}
$$

Note that in both cases, $X_3 = -1$.
If we do not know the value of Y_{n-1}, but we know the values of X_n, X_{n-1}, then we can recover the value of Y_{n-1} as $Y_{n-1} = |X_n - X_{n-1}|$.

Markov processes

Hence we can compute the conditional probability as

$$P(X_4 = k | X_3 = -1, X_2 = 3, X_1 = 1, X_0 = 0) = \begin{cases} p & k = 0 \\ 1 - p & k = -2 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X_4 = k | X_3 = -1, X_2 = -2, X_1 = -1, X_0 = 0) = \begin{cases} p & k = 1 \\ 1 - p & k = -3 \\ 0 & \text{otherwi} \end{cases}$$

If $(X_n)_{n \geq 0}$ is a Markov process, then following identities should hold true

$$\begin{aligned} & P(X_4 = k | X_3 = -1, X_2 = 3, X_1 = 1, X_0 = 0) \\ & = P(X_4 = k | X_3 = -1) \\ & = P(X_4 = k | X_3 = -1, X_2 = -2, X_1 = -1, X_0 = 0) \end{aligned}$$

but it fails.

Example random dynamics

As a model of (random) discrete time dynamics, we set $V_n = X_n - X_{n-1}$, which is a velocity of X. We consider a random dynamics with following difference equation;

$$\begin{aligned} X_{n+1} &= X_n + f(X_n, V_n) + \text{random term} \\ &= X_n + f(X_n, X_n - X_{n-1}) + \text{random term} \end{aligned}$$

where random term depends on X_n, X_{n-1}. Then this process $(X_n)_{n\geq 0}$ is not a Markov process. But if we set $Y_n := (X_n, V_n) = (X_n, X_n - X_{n-1})$ Then, it is easy to see that $(Y_n)_{n\geq 0}$ is a Markov process.